Micro Economic Theory II

Lecture Notes

(Notes are based on various books, lecture notes etc.)

Spring Semester 2019
Ashoka University

Dyotona Dasgupta

April 24, 2019

Contents

Chapter 1

Introduction

In Microeconomic Theory II, we will study game theory, its applications and various forms of *market failure*. Market failure is a situation where market equilibria fail to be Pareto optimal.

We can divide this course broadly into two parts. In Part I, we will study different concepts of *non-cooperative* game theory. This will help us understand different cases of market failure which we will study in Part II. We will start Part II with the settings in which some agents in the economy have *market power*. We will see that market equilibria fail to be Pareto optimal here. Next we will study *externalities* and *public goods*. In both the cases, the actions of one agent directly affect the utility functions of the other agents in the economy. There we will see that the presence of these nonmarketed "goods" or "bads" undermine Pareto optimality of market equilibrium. In the end, we will consider situations in which *asymmetry of information* exists among market participants and will see this informational asymmetry gives rise to welfare loss.

Part I

Game Theory

Game theory is the method of systematic study of situations of interactive decision making, using mathematical tools. These are situations involving several decision makers (called players) with different goals, in which the decision of each affects the outcome for all the decision makers. This interactivity distinguishes game theory from standard decision theory, which involves a single decision maker, and it is its main focus. Game theory tries to predict the behavior of the players and sometimes also provides decision makers with suggestions regarding ways in which they can achieve their goals.

The applicability of game theory is due to the fact that it is a context-free mathematical toolbox that can be used in any situation of interactive decision making. Consider for example a situation where you are deciding whether to carry an umbrella or not. This is not a strategic decision as the outcome does not depend on any other's action. Now, think of a situation where you are trying to cross a busy road. You are thinking whether to let go a car or cross before that, this is a strategic decision as the outcome (whether there will be an accident or not) not only depends on your action but also on the driver's action.

Game theory has huge applications in almost all areas including our daily life, to give you an idea the range of situations to which game theory can be applied: firms competing for business, political candidates competing for votes, jury members deciding on a verdict, animals fighting over prey, bidders competing in an auction, the evolution of siblings' behaviour towards each other, competing experts' incentives to provide correct diagnoses, legislators' voting behavior under pressure from interest groups, and the role of threats and punishment in long-term relationships. In this course, we are interested in applications of game theory in various economic problems – we will study how economic agents interact strategically. We will find out the outcomes and compare them with the (Pareto) efficient outcomes. We will see in most of the cases the outcomes will not be Pareto efficient, we will try to understand the reason behind this observation.

In Part I we will study noncooperative game theory mostly when all the players have complete information about the game – each player knows the strategies available to all the players and their payoffs. We will start with the analysis of simultaneous move game – the players move independently without observing actions taken by the other players. After that we will consider the situation where players move sequentially. We will end Part I with a brief discussion on games with incomplete information.

Chapter 2

Static Games of Complete Information

2.1 Strategic-Form Game

Definition 2.1. *A game in strategic (normal) form is a triple (has three elements)* $\mathcal{G} \equiv \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$*:*

(a) The set of players $i \in N = \{1, ..., n\}$*. We assume* N *to be finite,*

(b) The strategies available to each player. S_i *is the set of strategies of player* i*, for every player* $i \in N$ *– the set of strategy profiles is denoted as* $S \equiv S_1 \times ... \times S_n$*,*

(c) Payoff function or utility function u_i *that associates with each profile of strategies* $s \equiv (s_1, ..., s_n)$*, a payoff* $u_i(s)$ *for every player* $i \in N$*.*

Here, the set of strategies can be finite or infinite. When the S_i is finite for each $i \in N$, we refer that game as a finite game. The assumption is that players choose these strategies simultaneously in the game. This does not imply that the players necessarily choose their strategies at the same time – this just means that they choose their strategies independently without observing each others strategies.

A *strategy profile* of all the players will be denoted as $s \equiv (s_1, ..., s_n) \in S$. A strategy profile of all the players excluding player i will be denoted by s_{-i}. The set of all strategy profiles of players other than a player i will be denoted by S_{-i}.

A finite strategic game in which there are two players can be described conveniently in a table like that in figure 2.1. One player's actions are identified with the rows ("row player") and the other player's with the columns ("column player"). If the players' names are "1" and "2" then the convention is that the row player is player 1 and the column player is player 2. Thus in the game in figure 2.1 the set of actions of player 1 is $\{T, B\}$ and that of the player 2 is $\{L, R\}$. Now, the two numbers in each box formed by row r and column c are the players' payoffs when the row player chooses r and the column player chooses c. So, in the game in figure 2.1 player 1's payoff from the outcome (T, L) is w_1 and player 2's payoff is w_2.

		Player 2	
		L	R
Player 1	T	w_1, w_2	x_1, x_2
	B	y_1, y_2	z_1, z_2

Figure 2.1: A convenient representation of a two-player strategic game in which each player has two actions.

Let us consider some examples to illustrate games in strategic form.

Example 2.1. (Grading game) *Two students choose between "α" and "β" simultaneously. The grading policy is as follows:*

(a) If both of them choose α, then they get B^-.

(b) If both of them choose β, then they get B^+.

(c) If one of them chooses α and the other student chooses β, then the student who chooses α gets A and the other student gets C.

The question we ask is – how to represent this situation as a strategic-form game? In any strategic form game there are three elements: (a) players, (b) strategies of players and (c) their payoffs. So in order to represent the above situation as a game, we need to know the payoffs of each player. Now player's payoffs may depend on various things, in the above situation for example, a student may care only about his own grade or may care about his own as well his partner's payoff and so on.

Example. 2.1.1. *Suppose both the players care only about his own grade: Payoff from getting A, B^+, B^- and C are $3, 1, 0$ and -1 respectively.*

Now we have all the three elements to represent it as a normal-form game:

(a) Players: Student 1, Student 2.

(b) Strategies: Each student has two strategies – "α" and "β".
So, $S_1 = \{\alpha, \beta\}$ and $S_2 = \{\alpha, \beta\}$

(c) Payoff of student 1 is

$$u_1(s_1, s_2) = \begin{cases} 0 & \text{if } s_1 = s_2 = \alpha \\ 3 & \text{if } s_1 = \alpha \text{ and } s_2 = \beta \\ -1 & \text{if } s_1 = \beta \text{ and } s_2 = \alpha \\ 1 & \text{if } s_1 = s_2 = \beta. \end{cases}$$

Similarly payoff of student 2 is

$$u_2(s_1, s_2) = \begin{cases} 0 & \text{if } \; s_1 = s_2 = \alpha \\ -1 & \text{if } \; s_1 = \alpha \text{ and } s_2 = \beta \\ 3 & \text{if } \; s_1 = \beta \text{ and } s_2 = \alpha \\ 1 & \text{if } \; s_1 = s_2 = \beta. \end{cases}$$

We can conveniently represent it in table 2.2

		Student 2	
		α	β
Student 1	α	$0, 0$	$3, -1$
	β	$-1, 3$	$1, 1$

Figure 2.2: Grading Game: Selfish Players

Example. 2.1.2. (Variant of Grading Game) *Each student cares not only about his own grade but also about the grade of the other student: He likes getting an A but he feels guilty that this is at the expense of his pair getting a C. The guilt lowers his payoff from 3 to −1.*

We represent it as a normal-form game:

(a) Players: Student 1, Student 2.

(b) Strategies: Each student has two strategies – "α" and "β".
So, $S_1 = \{\alpha, \beta\}$ and $S_2 = \{\alpha, \beta\}$

(c) Payoff of student 1 is

$$u_1(s_1, s_2) = \begin{cases} 0 & \text{if } \; s_1 = s_2 = \alpha \\ -1 & \text{if } \; s_1 = \alpha \text{ and } s_2 = \beta \\ -1 & \text{if } \; s_1 = \beta \text{ and } s_2 = \alpha \\ 1 & \text{if } \; s_1 = s_2 = \beta. \end{cases}$$

Similarly payoff of student 2 is

$$u_2(s_1, s_2) = \begin{cases} 0 & \text{if } \; s_1 = s_2 = \alpha \\ -1 & \text{if } \; s_1 = \alpha \text{ and } s_2 = \beta \\ -1 & \text{if } \; s_1 = \beta \text{ and } s_2 = \alpha \\ 1 & \text{if } \; s_1 = s_2 = \beta. \end{cases}$$

		Student 2	
		α	β
Student 1	α	$0,0$	$-1,-1$
	β	$-1,-1$	$1,1$

Figure 2.3: Variant of Grading Game

So, the payoff matrix looks like figure 2.3.

Example 2.2. (Prisoner's Dilemma) *Two people are arrested for a crime. The police lack sufficient evidence to convict either suspect and consequently need them to give testimony against each other. The police put each suspect in a different cell to prevent the two suspects from communicating with each other. The police tell each suspect that if he testifies against (doesn't cooperate with) the other, he will be released and will receive a reward for testifying, provided the other suspect does not testify against him. If neither suspect testifies, both will be released on account of insufficient evidence and no rewards will be paid. If one testifies, the other will go to prison; if both testify, both will go to prison, but they will still collect rewards for testifying.*

Suppose, both the prisoners care only about their own payoffs: If both players cooperate (C) (do not testify), they get 2 *each. If they both play noncooperatively (D, for defect)they obtain 0. If one cooperate and the other does not, the latter is rewarded (gets* 3*)and the former is punished (gets -1).*

We represent it as a normal-form game:

(a) Players: Prisoner 1, Prisoner 2.

(b) Strategies: Each prisoner has two strategies – "Cooperate (C)" and "Defect (D)". So, $S_i = \{C, D\}$, where $i = \{1, 2\}$.

(c) Payoff is

$$u_i(s_i, s_j) = \begin{cases} 2 & \text{if } s_i = s_j = C \\ -1 & \text{if } s_i = C \text{ and } s_j = D \\ 3 & \text{if } s_i = D \text{ and } s_j = C \\ 0 & \text{if } s_i = s_j = D. \end{cases}$$

The payoff-matrix is as follows

Example 2.2.1. (Variant of Prisoner's Dilemma – altruistic preferences) *Suppose, both of them care about each other: If both players cooperate (C) (do not testify), they get 1 each. If they both play noncooperatively (D, for defect) they obtain 0. If one cooperate and the other does not, the latter is rewarded but he feels bad such that his payoff is* 1 *and the former is punished (gets -1).*

		Player 2 C	Player 2 D
Player 1	C	$2, 2$	$-1, 3$
	D	$3, -1$	$0, 0$

Figure 2.4: Prisoner's Dilemma

We represent it as a normal-form game:

(a) Players: Prisoner 1, Prisoner 2.

(b) Strategies: Each prisoner has two strategies – "Cooperate (C)" and "Defect (D)". So, $S_i = \{C, D\}$, where $i = \{1, 2\}$.

(c) Payoff is

$$u_i(s_i, s_j) = \begin{cases} 2 & \text{if } s_i = s_j = C \\ -1 & \text{if } s_i = C \text{ and } s_j = D \\ 1 & \text{if } s_i = D \text{ and } s_j = C \\ 0 & \text{if } s_i = s_j = D. \end{cases}$$

The payoff-matrix is as follows

		Player 2 C	Player 2 D
Player 1	C	$2, 2$	$-1, 1$
	D	$1, -1$	$0, 0$

Figure 2.5: Variant of Prisoner's Dilemma – altruistic preferences

Example 2.3. (Battle of the Sexes) *The name of the game is derived from the following description. A couple is trying to plan what they will be doing on the weekend. The alternatives are watching a movie (M) or a football match (F). The man prefers movie and the woman prefers the match, but both prefer being together to being alone, even if that means agreeing to the less-preferred recreational pastime.*
If the woman ends up at the football game with him, her payoff is 2; if she ends up at the movie with him, her payoff is 1; and if she ends up at either place without him, her payoff is 0. Similarly for the man, if he ends up at the movie with her, his payoff is 2; if he ends up at the football game with her, his payoff is 1; and if he ends up at either place without her, his payoff is 0.

We represent it as a normal-form game:

(a) Players: Woman, Man.

(b) Strategies: Each player has two strategies – "Movie (M)" and "Football (F)". So, $S_{Woman} = \{M, F\}$ and $S_{Man} = \{M, F\}$.

(c) Payoff of the woman is

$$u_{\text{Woman}}(s_{Woman}, s_{Man}) = \begin{cases} 2 & \text{if } s_{Woman} = s_{Man} = F \\ 0 & \text{if } s_{Woman} = F \text{ and } s_{Man} = M \\ 0 & \text{if } s_{Woman} = M \text{ and } s_{Man} = F \\ 1 & \text{if } s_{Woman} = s_{Man} = M. \end{cases}$$

Similarly payoff of the man is

$$u_{\text{Man}}(s_{Woman}, s_{Man}) = \begin{cases} 1 & \text{if } s_{Woman} = s_{Man} = F \\ 0 & \text{if } s_{Woman} = F \text{ and } s_{Man} = M \\ 0 & \text{if } s_{Woman} = M \text{ and } s_{Man} = F \\ 2 & \text{if } s_{Woman} = s_{Man} = M. \end{cases}$$

The payoff-matrix is as follows

		Man	
		F	M
Woman	F	$2, 1$	$0, 0$
	M	$0, 0$	$1, 2$

Figure 2.6: Battle of Sexes

Example 2.4. (Voting Game) *Imagine there are two candidates. The candidates choose their political positions on a political spectrum for an election. For simplicity, we assume that this political spectrum has five positions*

1 2 3 4 5

Figure 2.7: Voting Game

Voters are uniformly distributed – there are 20% voters at each of these positions. Voters vote for the closest candidate: the candidate whose position is closest to their own. If there is a tie then the voters of that position split evenly. The utility of the candidate is the vote share.

We represent the game in normal form

(a) Players: Two candidates

(b) Strategy: $S_1 = \{1, 2, 3, 4, 5\}$ and $S_2 = \{1, 2, 3, 4, 5\}$

(c) Payoff: If location of candidate 1 is x_1 and that of candidate 2 is x_2, then utilities are:

$$u_1(x_1, x_2) = \begin{cases} 20 \cdot x_1 + 20 \cdot \dfrac{x_2 - x_1 - 1}{2} & \text{if } x_1 \leq x_2 \\ 50 & \text{if } x_1 = x_2 \\ 20 \cdot [5 - x_1 + 1] + 20 \cdot \dfrac{x_1 - x_2 - 1}{2} & \text{if } x_1 > x_2. \end{cases}$$

The objective of game theory is to provide predictions of games. We start our analysis with the strongest possible prediction of a game.

Relevant Parts of the Reference Book: Chapter 1 and 3 (*Classic Normal Form Games*).

2.2 Domination

2.2.1 Strict Dominance

Consider the Grading game (example 2.1.1), what should player 1 choose in this case? Comparing Player 1's strategies α and β we find that

- If Player 2 plays α, the payoff to Player 1 under strategy α is 0, compared to only -1 under strategy β.

- If Player 2 plays β, the payoff to Player 1 under strategy α is 3, compared to only 1 under strategy β.

So, we see that independently of whichever strategy is played by Player 2, strategy α always yields a higher payoff to Player 1 than strategy β. This motivates the following definition:

Definition 2.2. *A strategy s_i of player i is* **strictly dominated** *if there exists another strategy $s_i^{'}$ of player i such that for each strategy vector $s_{-i} \in S_{-i}$ of the other players,*

$$u_i(s_i, s_{-i}) < u(s_i^{'}, s_{-i}).$$

In this case, we say that s_i is strictly dominated by $s_i^{'}$, or that $s_i^{'}$ strictly dominates s_i.

In the Grading game strategy β is strictly dominated by strategy α. It is therefore reasonable to assume that if Player 1 is "rational", she will not choose β, because under any scenario in which she might consider selecting β, the strategy α would be a better choice. This is the first rationality property that we assume.

Assumption 2.1. *A rational player will not choose a strictly dominated strategy.*

We also assume that all the players are rational

Assumption 2.2. *All players in a game are rational.*

Observe in the Prisoner's Dilemma game (example 2.2), for both the players, strategy "Cooperate" is strictly dominated by the strategy "Defect".

The prisoner's dilemma illustrates one of the major tensions in strategic settings: the clash between individual and group interests. The players realize that they are jointly better off if they each select C rather than D. However, each has the individual incentive to defect by choosing D. Because the players select their strategies simultaneously and independently, individual incentives win. One can even imagine the players discussing at length the virtues of the (C, C) strategy profile, and they might even reach an oral agreement to play in accord with that profile. But when the players go their separate ways and submit their strategies individually, neither has the incentive to follow through on the agreement. Strong individual incentives can lead to group loss.

Suppose we wish to compare the outcomes induced by two strategy profiles, s and $s^{'}$. We do this comparison using *Pareto efficient* criterion. We say that s is more efficient than $s^{'}$ if all of the players prefer the outcome of s to the outcome of $s^{'}$ and if the preference is strict for at least one player. In mathematical terms, s is more efficient than $s^{'}$ if

$$u_i(s) \geq u_i(s^{'}) \;\; \forall i \qquad \text{and} \qquad u_i(s) \geq u_i(s^{'}) \;\; \text{for at least one } i.$$

A strategy profile s is called efficient if there is no other strategy profile that is more efficient; that is, there is no other strategy profile $s^{'}$ such that $u_i(s^{'}) \geq u_i(s)$ for every player i and$u_j(s^{'}) \geq u_j(s)$ for some player j.

Note that, in the prisoner's dilemma, (C, C) is more efficient than (D, D). However, as discussed earlier due to strong individual incentive this efficient outcome cannot be achieved.

What about the "Variant of Prisoner's Dilemma" (example 2.2.1) or "Battle of Sexes" (example 2.3) games? Are there any strictly dominated strategies?

Common Knowledge and Iterated Elimination of Strictly Dominated Strategies

Now, we ask the question – under assumptions 2.1 and 2.2, can a strictly dominated strategy be eliminated? Let us consider the following example to answer this question:

Example 2.5. *Player 1 has two strategies Top (T) and Bottom (B):* $S_1 = \{T, B\}$ *and player 2 has three strategies Left (L), Middle (M), Right (R). The payoff-matrix is given in figure 2.8*

Player 2

Player 1		L	M	R
	T	$1,0$	$1,2$	$0,1$
	B	$0,3$	$0,1$	$2,0$

Figure 2.8: Dominated Strategies

Strategy R of Player 2 is strictly dominated by strategy M, so given assumptions 2.1 and 2.2, player 2 will not play strategy R.

Can strategy R be eliminated, under these two assumptions? The answer is: not necessarily. It is true that if Player 2 is rational he will not choose strategy R, but if Player 1 does not know that Player 2 is rational, he is liable to believe that Player 2 may choose strategy R, in which case it would be in Player 1's interest to play strategy B. So, in order to eliminate the strictly dominated strategies one needs to postulate that:

- Player 1 knows that Player 2 is rational.

Player 2

Player 1		L	M	R
	T	$1,0$	$1,2$	$0,1$
	B	$0,3$	$0,1$	$2,0$

$\xrightarrow{\text{KR}}$

P 2

P 1		L	M	~~R~~
	T	$1,0$	$1,2$	~~$0,1$~~
	B	$0,3$	$0,1$	~~$2,0$~~

Now, in this truncated game, strategy B of Player 1 is strictly dominated by strategy T.[1] Can we eliminate this strategy? Yes, but for that we further need to assume that:

- Player 2 knows that Player 1 knows that Player 2 is rational.

- Player 2 knows that Player 1 is rational.

Player 2

Player 1		L	M
	T	$1,0$	$1,2$
	B	$0,3$	$0,1$

$\xrightarrow{\text{KKR}}$

P 2

P 1		L	M
	T	$1,0$	$1,2$
	~~B~~	~~$0,3$~~	~~$0,1$~~

[1]Note that in the original game, strategy B is not strictly dominated, it becomes that only after we eliminate strategy R of Player 2.

In this truncated game, strategy L of Player 2 is strictly dominated by strategy M.[2] In order to eliminate that we further need to assume that:

- Player 1 knows that Player 2 knows that Player 1 knows that Player 2 is rational.
- Player 1 knows that Player 2 knows that Player 1 is rational.

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\textbf{Player 2}} \\ & & L & M \\ \hline \textbf{Player 1} & T & 1,0 & 1,2 \\ \hline \end{array} \xrightarrow{\text{KKKR}} \begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\textbf{P 2}} \\ & & \cancel{L} & M \\ \hline \textbf{P 1} & T & \cancel{1,0} & 1,2 \\ \hline \end{array}$$

Hence, the only strategy profile surviving such elimination is (T, M). So, Player 1 will play T and Player 2 will play M and their payoffs will be 1 and 2 respectively.

The process we just described is called *iterated elimination of strictly dominated strategies.* As we observed, it requires more than rationality. If we want to be able to apply the process for an arbitrary number of steps, we need the following assumption

Assumption 2.3. *The fact that all players are rational is* **common knowledge (CKR)** *among the players.*

That is, we need to assume not only that all the payers are rational, but also all the players know that all the players are rational, and that all the players know that all the players know that all the players are rational, and so on, *ad infinitum.*
Also, note that everything is happening simultaneously, given our assumption of common knowledge, both the players are *thinking* and playing the strategies which give them the highest payoffs.
Let us consider some more examples:

Example 2.6. *Player 1 has three strategies Top (T), Middle (M) and Bottom (B):* $S_1 = \{T, M, B\}$ *and player 2 has three strategies Left (L), Centre (C), Right (R):* $S_2 = \{L, C, R\}$*. The payoff matrix is given in figure 2.9*

		Player 2		
		L	C	R
	T	2, 2	6, 1	1, 1
Player 1	M	1, 3	5, 5	9, 2
	B	0, 0	4, 2	8, 8

Figure 2.9: Dominated Strategies

[2]Note again that strategy L becomes a strictly dominated strategy, only after we eliminate strategy R of Player 2 and strategy B of Player 1.

Strategy B is strictly dominated by strategy M for Player 1. Player 1 is rational, so he will not play B. Player 2 knows that Player 1 is rational. Then, he can conclude that Player 1 will not play B ever. So, we eliminate B strategy.

		Player 2		
		L	C	R
Player 1	T	$2,2$	$6,1$	$1,1$
	M	$1,3$	$5,5$	$9,2$

In this truncated game, strategy R is strictly dominated by Strategy L. So, Player 2 will not play R. Player 1 knows that Player 2 is rational and Player 1 knows that Player 2 knows that Player 1 is rational. So, we can again eliminate strategy R.

		Player 2	
		L	C
Player 1	T	$2,2$	$6,1$
	M	$1,3$	$5,5$

In this truncated game, strategy M is strictly dominated by strategy T. Like before, given common knowledge, we can again eliminate strategy M.

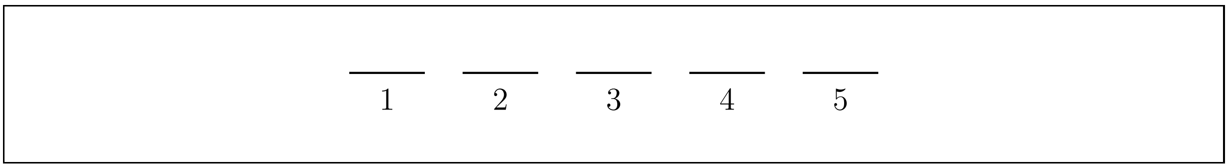
		Player 2	
		L	C
Player 1	T	$2,2$	$6,1$

Continuing in this manner, we will get that Player 2 does not play C. Hence, the only strategy profile surviving such elimination is (T, L).

		Player 2
		L
Player 1	T	$2,2$

So, Player 1 will play T and Player 2 will play L and their payoffs will be 2 and 2 respectively.

Recall the "Voting Game" (example 2.4) where the candidates choose their political positions to maximize their vote shares. Candidate i, where $i = \{1, 2\}$, has five strategies: $S_i = \{1, 2, 3, 4, 5\}$.



Voting Game

We want to check whether there is any strictly dominated strategy for any candidate or not.

1. We first argue that for candidate i strategy of choosing position 1 is strictly dominated by the strategy of choosing position 2.

   (i) If candidate j chooses position 1, then payoff of candidate i from choosing position 1 is 50, whereas his payoff from choosing position 2 is 80.

   (ii) If candidate j chooses position 2, then payoff of candidate i from choosing position 1 is 20, whereas his payoff from choosing position 2 is 50.

   (iii) If candidate j chooses position 3, then payoff of candidate i from choosing position 1 is 30, whereas his payoff from choosing position 2 is 40.

   (iv) If candidate j chooses position 4, then payoff of candidate i from choosing position 1 is 40, whereas his payoff from choosing position 2 is 50.

   (v) If candidate j chooses position 5, then payoff of candidate i from choosing position 1 is 50, whereas his payoff from choosing position 2 is 60.

   Similarly, it can also be shown that for candidate i strategy of choosing position 5 is strictly dominated by the strategy of choosing position 4.

   Now, both the candidates are assumed to be rational, so they will not choose any of these two positions. Moreover, each candidate knows that the other candidate is rational and hence he knows that the other candidate will not choose position 1 or position 5. So, these positions can be eliminated from the strategy spaces. So, we are left with

   $$\underset{2}{\underline{\qquad}} \quad \underset{3}{\underline{\qquad}} \quad \underset{4}{\underline{\qquad}}$$

2. In this truncated game, following the similar argument as above, it can be shown that for candidate i strategy of choosing position 2 is strictly dominated by the strategy of choosing position 3 and strategy of choosing position 4 is strictly dominated by the strategy of choosing position 3.

   Now, we need one more level of knowledge of rationality to eliminate strategy 2 and 4. We need to assume that the players are rational, each player knows that the other player is rational and moreover, each player knows that the other player knows that he is rational. Given our assumption of common knowledge, these are satisfied. So we eliminate 2 and 4.[3]

   $$\underset{3}{\underline{\qquad}}$$

   Hence, after iterated elimination of strictly dominated strategies, we are left with this unique strategy: Both the candidates choose the central position that is position 3.

[3] Note that in the original game, choosing position 2 and 4 are not strictly dominated, they become strictly dominated, only after we eliminate strategy 1 and 5.

In many games, iterated elimination of strictly dominated strategies lead to a unique outcome of the game. In those cases, we call it a solution of the game. A special case in which such a solution is guaranteed to exist is the family of games in which every player has a strategy that strictly dominates all of his other strategies, that is, a *strictly dominant strategy*.

Definition 2.3. *A strategy $s_i \in S_i$ for Player i is strictly dominant if for every $s_{-i} \in S_{-i}$, we have*

$$u_i(s_i, s_{-i}) > u_i(s_i^{'}, s_{-i}) \;\; \forall s_i^{'} \in S_i \setminus \{s_i\}.$$

Clearly, in that case, the elimination of all strictly dominated strategies leaves each player with only one strategy: his strictly dominant strategy. When this occurs we say that the game has a solution in strictly dominant strategies.

However, there are games in which iterated elimination of strictly dominated strategies does not yield a single strategy vector. For example, in a game that has no strictly dominated strategies, the process fails to eliminate any strategy. The game "Battle of Sexes" is one such example. We will see many more games as we go along in this course. Next, we will study a weak notion of dominance.

2.2.2 Weak Dominance

Example 2.7. *Player 1 has two strategies Top (T) and Bottom (B): $S_1 = \{T, B\}$ and Player 2 also has two strategies Left (L) and Right (R): $S_2 = \{L, R\}$. Payoffs are given in figure 2.10*

		Player 2	
		L	R
Player 1	T	$1,2$	$2,3$
	B	$2,2$	$2,0$

Figure 2.10: Weak Dominance

Comparing Player 1's strategies T and B we find that

- If Player 2 plays L, the payoff to Player 1 under strategy T is 1, compared to 2 under strategy B.
- If Player 2 plays R, the payoff to Player 1 under strategy T is 2, compared to 2 under strategy B.

So, we see that independently of whichever strategy is played by Player 2, strategy B grants him a payoff at least as high as strategy T yields, and if Player 2 chooses L, then strategy B provides strictly higher payoff. In this case, we say that strategy B *weakly dominates* strategy T (and strategy T is *weakly dominated* by strategy B). Accordingly, we define the following

Definition 2.4. *Strategy s_i of Player i is weakly dominated if there exists another strategy $s_i^{'}$ of Player i satisfying the following two conditions:*

(a) For every strategy vector $s_{-i} \in S_{-i}$ of the other players,

$$u_i(s_i, s_{-i}) \leq u_i(s_i^{'}, s_{-i}).$$

(b) There exists a strategy vector $s_{-i}^{'} \in S_{-i}$ of the other players such that

$$u_i(s_i, s_{-i}^{'}) < u_i(s_i^{'}, s_{-i}^{'}).$$

In this case we say that strategy s_i is weakly dominated by strategy $s_i^{'}$, and that strategy $s_i^{'}$ weakly dominates strategy s_i.

Clearly, strict domination implies weak domination. We also assume that a rational player does not play (*weakly* or strictly) dominated strategies.

Assumption 2.4. *A rational player will not choose a dominated strategy.*

Like before, we define *weakly dominant* strategy.

Definition 2.5. *A strategy $s_i \in S_i$ for Player i is weakly dominant if for every $s_{-i} \in S_{-i}$, we have*

$$u_i(s_i, s_{-i}) \geq u(s_i^{'}, s_i) \;\; \forall s_i^{'} \in S_i \setminus \{s_i\}.$$

In some games, weakly dominant strategies give striking prediction. One such example is Second -Price Sealed bid Auction/ Vickrey Auction. Though this example involves "incomplete" information which we will cover later in our course, but this helps us understand the power of "dominant" strategy.

2.2.2.1 Second-Price Sealed bid Auction/ Vickrey Auction

Example 2.8. (Second-Price Sealed bid Auction/ Vickrey Auction) *An indivisible object is offered for sale. There are two buyers. Player i has a valuation v_i for the object, that is if Player i gets the object and pays the price p, then his payoff is $v_i - p$. Each buyer i bids a price b_i (presented to the auctioneer in a sealed envelope). Bids are constrained to be nonnegative (observe, negative price means the auctioneer has to pay the buyer). The winner of the object is the buyer who makes the highest bid. That may not be surprising, but in contrast to the auctions most of us usually see, the winner does not proceed to pay the bid he submitted. Instead he pays the second-highest price offered (hence the name second-price auction). In case of a tie, the winner is determined by a flip of coin and he pays the second highest bid, which is also his own bid amount in this case.*

We first represent the Second -Price Sealed bid Auction/ Vickrey Auction as a normal-form game.

- Buyers are the players.
- The set of strategies available to buyer i is the set of the possible bids: $S_i = [0, \infty)$
- The payoff of buyer i, when the strategies are $b = (b_1, b_2)$, is

$$u_i(b) = \begin{cases} v_i - b_j & \text{if } b_i > b_j \\ \frac{1}{2}[v_i - b_j] & \text{if } b_i = b_j \\ 0 & \text{otherwise.} \end{cases}$$

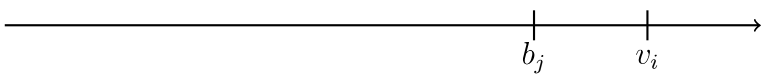
We show that in the Vickrey auction, it is a weakly dominant strategy for every buyer to bid his value.
We divide the set of strategies available to him $S_i = [0, \infty)$ into three subsets:

(a) The strategy in which his bid is equal to his true value: $b_i = v_i$

(b) The strategies in which his bid is less than his true value: $b_i < v_i$

(c) The strategies in which his bid is higher than his true value: $b_i > v_i$

We now show that strategy $b_i = v_i$ dominates all the strategies in the other two subsets. Suppose buyer j bids an amount b_j.
Case 1. $v_i > b_j$



(a) $b_i = v_i$

The buyer wins the auction and his utility is $v_i - b_j > 0$.

(b) $b_i < v_i$: There can be three cases

(i) $v_i > b_i > b_j$

The buyer wins the auction and his utility is $v_i - b_j > 0$.

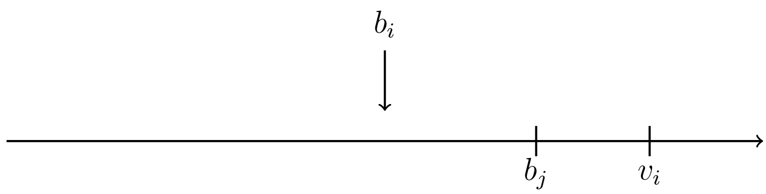
So in this case, utility from bidding less than the true value is the same as that from bidding the true value.

(ii) $v_i > b_i = b_j$

The buyer gets the object with probability $\frac{1}{2}$, his utility then is $\frac{1}{2}[v_i - b_j] < v_i - b_j$.

Hence, in this case, utility from bidding less than the true value is strictly less than that from bidding the true value.

(iii) $v_i > b_j > b_i$



The buyer does not get the object and his utility is 0. Hence, in this case, utility from bidding less than the true value is strictly less than that from bidding the true value.

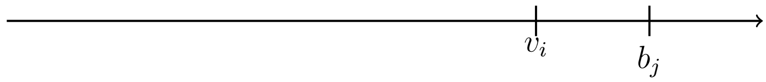
(c) $b_i > v_i$

The buyer wins the auction and his utility is $v_i - b_j > 0$.

So in this case, utility from bidding higher than the true value is the same as that from bidding the true value.

Hence, bidding v_i is a weakly dominant strategy for buyer i.

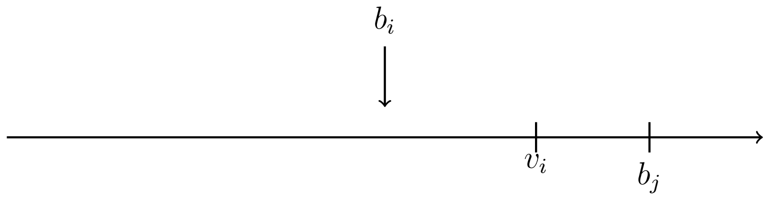
Case 2. $v_i \leq b_j$



(a) $b_i = v_i \leq b_j$

If $b_i = v_i < b_j$, the buyer does not get the object. If $b_i = v_i = b_j$, he gets the object with probability $\frac{1}{2}$. In both the cases, his utility is zero.

(b) $b_i < v_i$



The buyer does not get the object and his utility is zero.

So in this case, utility from bidding less than the true value is the same as that from bidding the true value.

(c) $b_i > v_i$: There can be three cases

(i) $v_i \leq b_j < b_i$

The buyer wins the auction and his utility is $v_i - b_j < 0$.

So in this case, utility from bidding higher than the true value is lower than that from bidding the true value.

(ii) $v_i \leq b_j = b_i$

The buyer gets the object with probability $\frac{1}{2}$, his utility then is $\frac{1}{2}[v_i - b_j] < 0$. Hence, in this case, utility from bidding higher than the true value is strictly less than that from bidding the true value.

(iii) $v_i < b_i < b_j$

The buyer does not get the object and his utility is 0. Hence, in this case, the utility from bidding higher than the true value is same as that from bidding the true value.

Hence, bidding v_i is a weakly dominant strategy for buyer i.
There is no foundation for eliminating (iteratively or otherwise) weakly dominated strategies, as if we remove weakly dominated strategies iteratively, then the order of elimination matters.

Example 2.9. *Player 1 has three strategies – Top (T), Middle (M) and Bottom (B):* $S_1 = \{T, M, B\}$ *and Player 2 has three strategies – Left (L), Centre (C) and Right (R):* $S_2 = \{L, C, R\}$. *The payoffs are as follows:*

		Player 2		
		L	C	R
	T	$1, 2$	$2, 3$	$0, 3$
Player 1	M	$2, 2$	$2, 1$	$3, 2$
	B	$2, 1$	$0, 0$	$1, 0$

Figure 2.11: Weakly Dominated Strategies – Order of Elimination Matters

There are two weakly dominated strategies for Player 1: $\{T, B\}$.

- Suppose Player 1 eliminates T first. Then, strategies $\{C, R\}$ are weakly dominated for Player 2. Suppose Player 2 eliminates R. Then, Player 1 eliminates the weakly dominated strategy B. Finally, Player 2 eliminates Strategy C to leave us with (M, L).

- Suppose Player 1 eliminates T first. Then, strategies $\{C, R\}$ are weakly dominated for Player 2. Suppose Player 2 eliminates C and then R. Then, Player 1 eliminates M. So, we are left with (B, L).

- Suppose Player 1 eliminates B first. Then, both strategies $\{L, C\}$ are weakly dominated for Player 2. Suppose Player 2 eliminates L first. Then, T is weakly dominated for Player 1. Eliminating T, we see that C is weakly dominated for Player 2. So, we are left with (M, R).

We summarize this in table 2.12

	Order of elimination from left to right	Result	Payoff
(1)	T, R, B, C	(M, L)	$(2, 2)$
(2)	T, C, R, M	(B, L)	$(2, 1)$
(3)	B, L, C, T	(M, R)	$(3, 2)$

Figure 2.12: Weakly Dominated Strategies – Order of Elimination Matters

Exercise 2.1. *Suppose that there are two workers, $i = 1, 2$, and each can "work" ($s_i = 1$) or "shirk" ($s_i = 0$). The total output of the team is $4(s_1+s_2)$ and is shared equally between the two workers. Each worker incurs private cost 3 while working and 0 while shirking. Write down this as a normal-form game. Check whether there is any dominated strategy or not. Does iterated elimination of strictly dominated strategy give us any solution?*

Exercise 2.2. *Following are the examples of some classic games (example 2.10 through example 2.13). Represent them in normal-form game. Check whether there is any strictly dominated strategy. Can you solve them using "iterated elimination of strictly dominated strategies"?*

Example 2.10. (Matching Pennies) *Each of two people chooses either Head or Tail. If the choices differ, person 1 pays person 2 a dollar; if they are the same, person 2 pays person 1 a dollar. Each person cares only about the amount of money that he receives.*

Example 2.11. (Hawk-Dove/Chicken) *Two animals are fighting over some prey. Each can behave like a dove or like a hawk. The best outcome for each animal is that in which it acts like a hawk while the other acts like a dove; the worst outcome is that in which both animals act like hawks. Each animal prefers to be hawkish if its opponent is dovish and dovish if its opponent is hawkish.*

Suppose if an animal acts like a hawk while the other acts like a dove, then its payoff be 4. If it acts like a dove while the other acts like a hawk, then its payoff be 3. If both act like doves then each of them gets 3 whereas if both act like hawks then each of them gets 0.

Example 2.12. (Coordination Game) *Recall the "Battle of Sexes" game (example 2.3). Suppose both of them like football and movie equally – each gets* 2 *if they watch the movie or the football match together.*

Example 2.13. (Pareto Coordination Game) *As in BoS (example 2.3), two people wish to go out together, but in this case they agree on the more desirable outing: Both of them prefer football over movies and get* 2 *each if they watch the football match, whereas if they watch the movie they get* 1 *each.*

In many games, players have more than one undominated strategies. For example consider the following games:

Example 2.14. $S_1 = \{U, M, D\}$ *and* $S_2 = \{L, C, R\}$. *The payoffs are given in figure 2.13*

		Player 2		
		L	C	R
	T	$8,3$	$0,4$	$4,4$
Player 1	M	$4,2$	$6,5$	$8,3$
	B	$3,7$	$5,1$	$4,0$

Figure 2.13: Players have More Than One Undominated Strategies

Observe, strategy B of player 1 is strictly dominated by strategy M, but no other strategy is dominated, that is all other strategies are undominated.

So, one cannot predict how people should or will play these games on the basis of the dominance criterion. We must move on to explore the process by which players actually select their strategies, at least among those that are not dominated.

Relevant Parts of the Reference Book:

> Chapter 5 (*Rationality, Common Knowledge*), 6 (*till "The Concept of Efficiency", "Weak Dominance"*), 7, 8. (While reading the book, skip the concepts involving "mixed strategy" for now and come back after we cover them in class, that is after 13^{th} February).

2.3 Belief and Expected Payoff

2.3.1 Belief

In game theory, players payoffs not only depend on their own actions but also on the actions of their opponents. For example in the coordination game (example 2.12), if the man (Player 2) goes to movie, then payoff of the woman (Player 1) from going to movie is higher than that from going to the football match. Conversely, Player 1's utility from choosing "football" is higher than that from choosing "movie" when Player 2 chooses "movie".

So, in games, it is wise to form an opinion about the other players' behaviour before deciding your own strategy. Rational players (who want to maximize their payoffs) indeed think about the actions that the other players might take. We use the term *belief* for a player's assessment about the strategies of the others in the game.

Definition 2.6. *A* **belief** *of player i is a probability distribution over the strategies of the other players. Let us denote such a probability distribution $\theta_i \in \Delta S_{-i}$,*[4] *where ΔS_{-i} is the set of probability distributions over the strategies of all the players except player i.*

2.3.2 Expected Payoff

Given these beliefs, a player computes his *expected payoff/utility*. Suppose, Player i's belief is θ_i, then his expected utility from playing strategy s_i is given by

$$u_i(s_i; \theta_i) := \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})\theta_i(s_{-i}) \quad \forall s_i \in S_i$$

Let us try to understand with an example.

Consider the game "Variant of Prisoner's Dilemma" (example 2.2.1). Suppose, Player 1 believes that Player 2 will play C with probability $\frac{1}{4}$ and D with probability $\frac{3}{4}$, that is $\theta_1(C) = \frac{1}{4}$ and $\theta_1(D) = \frac{3}{4}$, we can also write this belief as $\theta_1 = (\frac{1}{4}, \frac{3}{4})$. Then his expected payoff from playing C is

$$u_1(C; \theta_1) = \frac{1}{4} \cdot 2 + \frac{3}{4} \cdot (-1) = -\frac{1}{4}$$

And, his expected payoff from playing D is

$$u_1(D; \theta_1) = \frac{1}{4} \cdot 1 + \frac{3}{4} \cdot 0 = \frac{1}{4}$$

Now, $\theta_1(C)$ can take any value between 0 and 1 such that $\theta_1(C) + \theta_1(D) = 1$. We plot the expected payoffs of Player 1 for all beliefs in figure 2.14.

Discussion. Recall, the definitions of (strictly or weakly) dominant strategies (definitions 2.3 and 2.5) – irrespective of beliefs of Player i, playing $s_i^{'}$ is worse than playing s_i. In other words, a strategy s_i for Player i is dominant if and only if that is dominant for *all* plausible beliefs of Player i. (We will not prove this formally, but check this with an example). So, the idea of domination *does not* depend on players' beliefs.

[4] Observe θ_i is the belief of Player i about other players' strategies (s_{-i}).

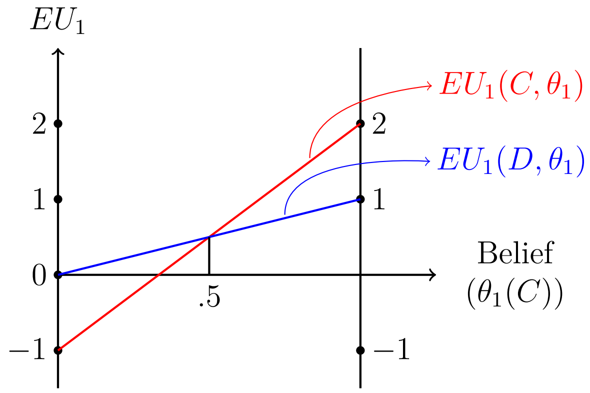


Figure 2.14: Belief and Expected Utility

2.4 Best Response

Given his belief, a rational player chooses strategies to maximize his expected payoff. Such a strategy is called a best response (or best reply).

Definition 2.7. *Suppose Player i has a belief $\theta_i \in \Delta S_{-i}$ about the strategies played by the other players. Player i's strategy s_i is a* **best response** *if*

$$u_i(s_i;\theta_i) \geq u_i(s_i^{'}, \theta_i) \ \ \forall s_i^{'} \in S_i.$$

For any belief θ_i of Player i, we denote the set of best responses by $BR_i(\theta_i)$.
In the game "Variant of Prisoner's Dilemma" (example 2.2.1), if Player 1's belief is $\theta_1 = (\frac{1}{4}, \frac{3}{4})$, then his best response is to play D, that is $BR_1\big((\frac{1}{4}, \frac{3}{4})\big) = \{D\}$, whereas if $\theta_1 = (\frac{3}{4}, \frac{1}{4})$, then his best response is to play C, so $BR_1\big((\frac{3}{4}, \frac{1}{4})\big) = \{C\}$.

Now let us consider some other examples.

Example 2.15. *Player 1 has three strategies: $S_1 = \{T, M, B\}$ and Player 2 has two strategies $\{L, R\}$. Payoffs are given in figure 2.15.*

We draw expected utilities of Player 1 from playing each of three strategies for all possible beliefs in figure 2.15. Hence, in this example when Player 1 belies that the probability of Player 2 playing L is lower than $\frac{1}{3}$, his best response is to play strategy B, whereas if that belief is exactly equal to $\frac{1}{3}$, he has two best responses – strategy B and strategy M. Below we write down Player 1's best responses for different beliefs. As you can see all three strategies of Player 1 are best response to some of his belief.

Player 2

		L	R
Player 1	T	$3, 5$	$-1, 3$
	M	$1, 2$	$2, 4$
	B	$-1, 1$	$3, 1$

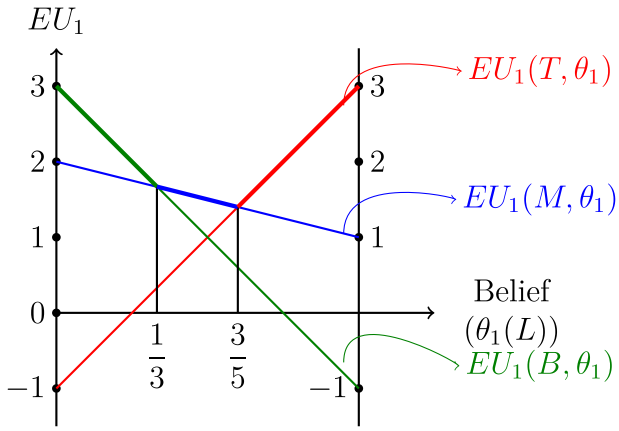


Figure 2.15: Belief, Expected Utility and Best Response

$$BR_1(\theta_1) = \begin{cases} \{B\} & \text{when } \theta_1(L) \in [0, \frac{1}{3}) \\ \{M, B\} & \text{when } \theta_1(L) = \frac{1}{3} \\ \{M\} & \text{when } \theta_1(L) \in (\frac{1}{3}, \frac{3}{5}) \\ \{T, M\} & \text{when } \theta_1(L) = \frac{3}{5} \\ \{T\} & \text{when } \theta_1(L) \in (\frac{3}{5}, 1]. \end{cases}$$

In the next example, one strategy (B) is not a best response to any belief. We call such a strategy **never-best response**. We define it formally below.

Definition 2.8. *A strategy of Player i in a strategic game is a* **never-best response** *if it is not a best response to any belief of Player i.*

A rational player who wants to maximize his own payoff never plays a "never-best response" strategy, as for any belief he can increase his payoff by playing best response to that belief.

Example 2.16. *Player 1 again has three strategies $S_1 = \{T, M, B\}$. Payoffs are given in the following payoff matrix (figure 2.16).*

In this example if Player 1 believes that the probability of Player 2 playing playing L is lower than $\frac{5}{8}$, his best response is to play strategy M, whereas if that belief is higher than $\frac{5}{8}$, his best response is to play strategy T. When that belief is exactly equal to $\frac{5}{8}$,

		Player 2	
		L	R
Player 1	T	$7,2$	$1,3$
	M	$4,6$	$6,3$
	B	$5,3$	$3,6$

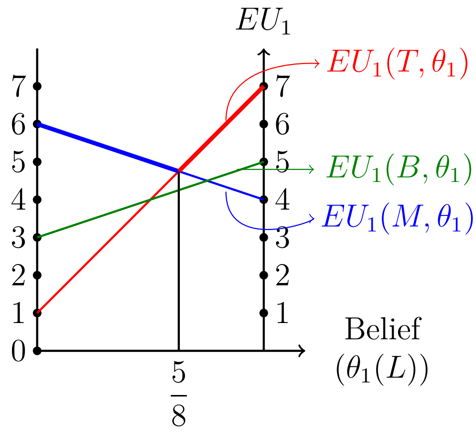


Figure 2.16: Belief, Expected Utility and Never-Best Response

he has two best responses – strategy T and strategy M. Below we write down Player 1's best responses for different beliefs.

$$BR_1(\theta_1) = \begin{cases} \{M\} & \text{when } \theta_1(L) \in [0, \frac{5}{8}) \\ \\ \{T, M\} & \text{when } \theta_1(L) = \frac{5}{8} \\ \\ \{T\} & \text{when } \theta_1(L) \in (\frac{5}{8}, 1]. \end{cases}$$

Observe, there does not exist belief of Player 1 such that strategy B is a best response to that belief.

How far can we go with this observation that a rational player never plays a never-best response strategy? In the next application we will see that this can give us a unique solution. This is an important example in Economics, and we will study this in more details in Part II.

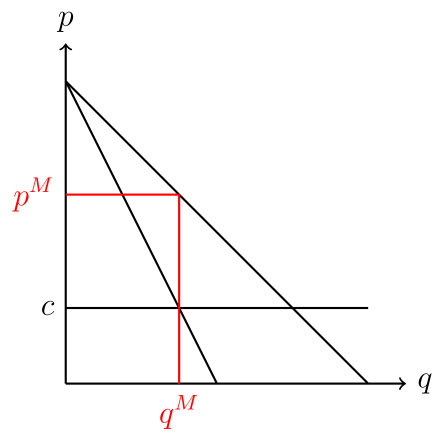
2.4.1 An Application

Suppose inverse demand function is $p(q) = a - bq$, where $a, b > 0$ and q is the total output. Cost of producing one unit of output is c. There is no fixed cost. There are n identical firms. We are interested in finding out the equilibrium quantity and price of the product.

Monopoly. First suppose that there is only one firm in the market ($n = 1$). The problem of the firm is to

$$\max_{q \geq 0} \Pi \equiv p \cdot q - c \cdot q \qquad \Leftrightarrow \max_{q \geq 0} q[a - bq] - cq$$

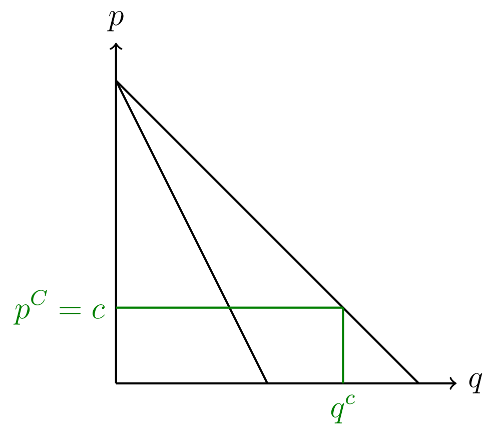
The first order condition (F.O.C.) is $\dfrac{\partial \Pi}{\partial q} = 0 \quad \Rightarrow \quad a - 2bq - c = 0 \quad \Rightarrow q = \dfrac{a-c}{2b}$.
In order to ensure that this quantity indeed maximizes the firm's profit (and not minimizes it) we need to check the second order condition.
Second order condition (S.O.C.) $\dfrac{\partial^2 \Pi}{\partial q^2} = -2b < 0$ (so S.O.C. is satisfied).



$q^M = \dfrac{a-c}{2b}$ and $p^M = \dfrac{a+c}{2}$.

Figure 2.17: Monopoly: Equilibrium Quantity and Price

Perfect Competition. There are (infinitely) many firms and each firm is infinitesimally small. So, no firm can influence the market price, each firm takes market price as given and chooses quantity to maximize profit.



$q^C = \dfrac{a-c}{b}$ and $p^C = c$.

Figure 2.18: Perfect Competition: Equilibrium Quantity and Price

Observe in both the cases, the firms are not strategic: In the monopolistic market there is only one firm and in the competitive market there are so many firms and each firm is so infinitesimally small that it takes market price as given.

Nest we study a market structure where there are a few firms, so they engage in strategic interaction – one firm's decision affects other firms' profits. The firms either choose quantities of production or price they want to charge. We start our analysis with the case where firms engage in quantity competition.

Duopoly: Cournot (Quantity Competition). There are two firms in the market. Cost functions are the same: marginal cost is c and there is no fixed cost of production. Each firm simultaneously and noncooperatively chooses the "quantity" of production to maximize its profit. Suppose firm 1 chooses to produce q_1 and firm 2 chooses to produce q_2, then from the inverse demand function we know that market price would be $p = a - bq = a - b(q_1 + q_2)$.

> This model was first published in 1838 by Antoine Augustin Cournot, and is the earliest predecessor of modern game theory. Cournot's work is one of the classics of Game Theory.

Let us first write down this as a normal form game.

- There are two players: Firm 1 and Firm 2.
- Each firm's strategy space, in this case which is quantity, can be represented as $q_i = [0, \infty)$, where $i = \{1, 2\}$.[5]
- We assume that the firm's payoff is simply its profit. Thus, the payoff of firm i is

$$\Pi_i(q_i, q_j) = p(q) \cdot q_i - c \cdot q_i = q_i[a - b(q_i + q_j) - c]$$

Now consider the problem of firm 1. It chooses quantity to maximize its own profit, given its belief about firm 2's choice. So, the problem of firm 1 is to

$$\max_{q_1 \geq 0} q_1[a - b(q_1 + q_2) - c]$$

The first order condition gives us

$$\text{F.O.C.:} \quad \frac{\partial \Pi}{\partial q_1} = 0 \quad \Rightarrow q_1 = \frac{a-c}{2b} - \frac{q_2}{2}.$$

Again we check for the second order condition in order to ensure that this quantity indeed maximizes firm 1's profit.

$$\text{S.O.C.:} \quad \frac{\partial^2 \Pi}{\partial q_1^2} = -2b < 0 \ \ \text{(so, S.O.C. is satisfied).}$$

Hence, the best response of firm 1, when it believes that firm 2 produces q_2 is

$$BR_1(q_2) = \frac{a-c}{2b} - \frac{q_2}{2}$$

[5] Naturally, negative outputs are not feasible. One could argue that extremely large quantities are also not feasible and so should not be included in a firm's strategy space. But because $p(q) = 0$ for $q \geq a$, however, neither firm will produce a quantity $q_i > a$.

This is also known as *reaction function* of firm 1. Observe, $BR_1(q_2)$ is decreasing in q_2 (why?). We draw it in figure 2.19.

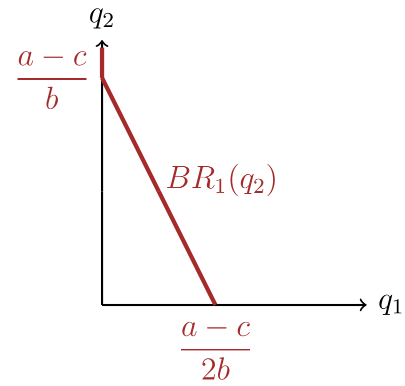


Figure 2.19: Cournot: Best Response Function of Firm 1

Consider the horizontal and vertical intercepts.
Horizontal Intercept. When Firm 1 believes that Firm 2 produces nothing, its best response is to produce the monopoly quantity, that is $q^M = \dfrac{a-c}{2b}$.
Vertical Intercept. When firm 2 produces $\dfrac{a-c}{b}$, that is the quantity produced in the competitive market, even if Firm 1 produces nothing the market price will be equal to the marginal cost. So, if Firm 1 produces any positive amount, price will decrease further and it will be lower than the marginal cost. Hence, if Firm 1 believes that Firm 2 produces no less than q^C, then it would produce nothing.

Now observe both the firms are identical,[6] so similarly firm 2's best response is given by

$$BR_2(q_1) = \frac{a-c}{2b} - \frac{q_1}{2}.$$

In figure 2.20 we superimpose the best response (reaction) functions of both the firms.

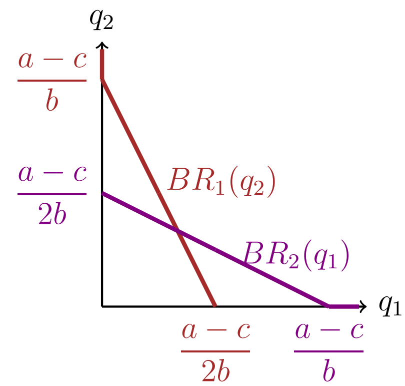


Figure 2.20: Cournot: Best Response Functions

[6]We call this kind of game *symmetric game*. We will define it formally later.

Now observe, any quantity more than $\frac{a-c}{2b}$ is never-best response for both the firms: $BR_1(q_2) \in [0, \frac{a-c}{2b}]$ and $BR_2(q_1) \in [0, \frac{a-c}{2b}]$. Given common knowledge we can eliminate those strategies and get figure 2.21.

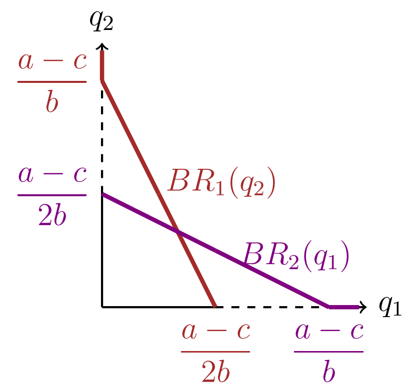


Figure 2.21: Cournot: First Round Elimination of Never-Best Responses

But, Firm 1's best response to produce $q_1 \in [0, \frac{a-c}{4b})$ is to beliefs about Firm 2's those strategies which are never-best response. So, Firm 1 never chooses $q_1 \in [0, \frac{a-c}{4b})$. Similarly, Firm 2 never chooses $q_2 \in [0, \frac{a-c}{4b})$. We draw this in figure 2.22.

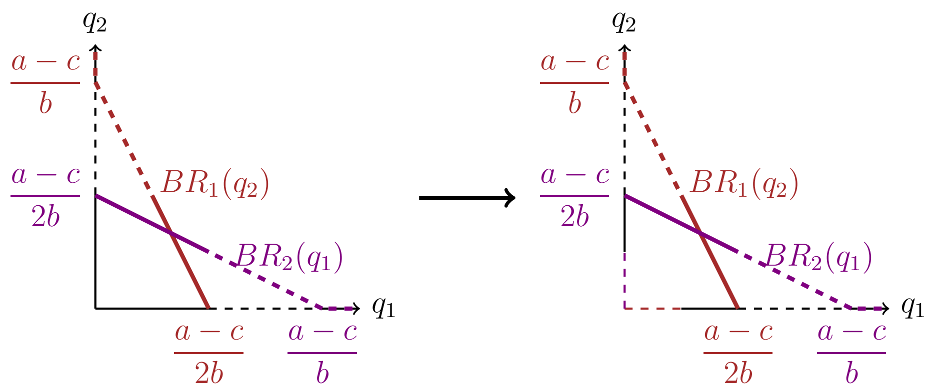


Figure 2.22: Cournot: Second Round Elimination of Never-Best Responses

If we keep on doing like this[7] we get that

$$q_1^* = q_2^* = \frac{a-c}{3b}. \quad \text{Hence, } q^O = q_1^* + q_2^* = \frac{2(a-c)}{3b} \text{ and } p^O = \frac{a+2c}{3}.$$

This is the unique equilibrium of this model. That is, no one has any incentive to deviate unilaterally. This stability property was defined by John Nash, who invented the

[7] Note that players are playing simultaneously, each firm is just thinking carefully how much to produce keeping in mind that the other firm is rational and hence will not play a never-best response strategy.

equilibrium concept that bears his name. We study **Nash Equilibrium** next. This is one of the most influential concepts in Game Theory and has huge applications, almost in every area.

Exercise 2.3. *Will the firms have a tendency to adjust their outputs if they are not in equilibrium initially?*
Hint: *Start with any quantity-price pair which are not at equilibrium. Show that the firms adjust from non-equilibrium to equilibrium. Draw picture.*

2.5 Nash Equilibrium

To motivate let us consider another game.

Example 2.17. *There are two players. Player 1 has three strategies Top (T), Middle (M) and Bottom (B). Player 2 also has three strategies Left (L), Centre (C) and Right (R). Payoffs are given in the following payoff matrix (figure 2.23).*

		Player 2		
		L	C	R
	T	$0, 6$	$6, 0$	$4, 3$
Player 1	M	$6, 0$	$0, 6$	$4, 3$
	B	$3, 3$	$3, 3$	$5, 5$

Figure 2.23: Nash Equilibrium – Motivation

In this game

- If Player 2 knows that Player 1 will choose T, he will choose L (best response to T).
- If Player 1 knows that Player 2 will choose L, he will choose M (best response to L).
- If Player 2 knows that Player 1 will choose M, he will choose C (best response to M).
- If Player 1 knows that Player 2 will choose C, he will choose T (best response to C).
- If Player 2 knows that Player 1 will choose B, he will choose R (best response to B).
- If Player 1 knows that Player 2 will choose R, he will choose B (best response to R).

In figure 2.24 we underline the payoffs from playing the best responses.

The pair of strategies (B, R) is a Nash equilibrium as each strategy in this pair is the best response to the other strategy. In the figure 2.24 that corresponds to the cell where both the payoffs are underlined. Alternatively, we can state that (B, R) is a Nash equilibrium because neither player has a **profitable unilateral deviation**; that

Player 2

		L	C	R
Player 1	T	$0, \underline{6}$	$\underline{6}, 0$	$4, 3$
	M	$\underline{6}, 0$	$0, \underline{6}$	$4, 3$
	B	$3, 3$	$3, 3$	$\underline{5}, \underline{5}$

Figure 2.24: Nash Equilibrium – Motivation

is, under the assumption that the other player indeed chooses his strategy according to (B, R), neither player has a strategy that grants a higher payoff than sticking to (B, R). Observe in this game (B, R) is the unique Nash equilibrium. Below we define Nash equilibrium formally.

Definition 2.9. *A strategy profile* $(s_1^*, ..., s_n^*)$ *is a* **Nash equilibrium** *if for all* $i \in N$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \ \ \forall s_i \in S_i.$$

The definition above requires that given strategies of other players s_{-i}^*, no player i has a **profitable unilateral deviation**. We say that a strategy $\hat{s}_i \in S_i$ is a profitable unilateral deviation of player i at a strategy vector $s \in S$ if $u_i(\hat{s}_i, s_{-i}) \geq u_i(s)$.
To understand it better we provide two more definitions of Nash equilibrium.

Definition. 2.9.1. (A Belief based Definition of Nash Equilibrium) *A strategy profile* $(s_1^*, ..., s_n^*)$ *is a Nash equilibrium if for all* $i \in N$

$$\theta_i(s_{-i}^*) = 1 \ \textit{and} \ u_i(s_i^*; \theta_i) \geq u_i(s_i; \theta_i) \ \ \forall s_i \in S_i.$$

This definition is saying that if Player i believes that all other players are playing s_{-i}^*, then s_i^* constitutes Nash equilibrium if utility from playing s_i^* is no less than utility from playing some other strategy. Otherwise, Player i had an incentive to deviate (to that strategy) unilaterally.

Definition. 2.9.2. (Definition of Nash Equilibrium using Best Response) *A strategy profile* $(s_1^*, ..., s_n^*)$ *is a Nash equilibrium if for all* $i \in N$

$$s_i^* \in BR_i(s_{-i}^*).$$

Let us consider some examples to understand the concept, we will start with games which we have already seen earlier in our course.
Grading Game: Consider example 2.1.1 The unique Nash equilibrium is (α, α) (see figure 2.25), in which both the students announce α, resulting in payoff $(0, 0)$. Recall that this is the same result that is obtained by elimination of strictly dominated strategies.[8]

[8] We will discuss the relation between Nash equilibrium and iterated elimination of strictly dominated strategies later in this section.

Student 2

		α	β
Student 1	α	$\underline{0}, \underline{0}$	$\underline{3}, -1$
	β	$-1, \underline{3}$	$1, 1$

Figure 2.25: Grading Game with Selfish Players: Nash Equilibrium

Prisoner's Dilemma: Consider example 2.2. The unique Nash equilibrium is (D, D) (see figure 2.26), in which both prisoners testify, resulting in payoff $(0, 0)$.

Player 2

		C	D
Player 1	C	$2, 2$	$-1, \underline{3}$
	D	$\underline{3}, -1$	$\underline{0}, \underline{0}$

Figure 2.26: Prisoner's Dilemma: Nash Equilibrium

We next address the relation between Nash equilibrium and iterated elimination of strictly dominated strategies. Observe that the Nash equilibrium of the Prisoner's Dilemma or the Grading are the only strategies that survive iterated elimination of strictly dominated strategies. We generalize this in the following claim.

Claim 2.1. *If $(s_1^*, ..., s_n^*)$ is a strictly dominant strategy equilibrium, it is a unique Nash equilibrium.*

Proof. If s_i^* is a strictly dominant strategy of Player i, then $\{s_i^*\} = BR_i(s_{-i})$ for all $s_{-i} \in S_{-i}$. This is true for all $i \in \{1, .., n\}$. ■

Next we observe that Nash equilibrium is a stronger *solution concept* than iterated elimination of strictly dominated strategies in the following sense. If the strategies $(s_1^*, ..., s_n^*)$ are a Nash equilibrium then they survive iterated elimination of strictly dominated strategies, but there can be strategies that survive iterated elimination of strictly dominated strategies but are not part of any Nash equilibrium.

Exercise 2.4. *Show that "if the strategies (s_1^*, s_2^*) are a Nash equilibrium then they survive iterated elimination of strictly dominated strategies."*
Hint. *Prove this by contradiction, that is suppose (s_1^*, s_2^*) are a Nash equilibrium but at least one of them get eliminated by iterated elimination of strictly dominated strategies. Now write the definition of strictly dominated strategy and Nash equilibrium in this context and you will find a contradiction.*

Exercise 2.5. *Can you generalize this claim to any finite n?*

Exercise 2.6. *Provide an example of a game such that in that game there are strategies which survive iterated elimination of strictly dominated strategies but are not part of any Nash equilibrium.*

The question we now ask is – Can we say the same for weakly dominated strategies? That is can we say that if the strategies $(s_1^*, ..., s_n^*)$ are a Nash equilibrium then they cannot be weakly dominated? The answer turns out to be no. In the following example we show that in the unique Nash equilibrium both the players play weakly dominated strategies.

Example 2.18. *Player 1 has three strategies* $S_1 = \{T, M, B\}$ *and Player 2 has three strategies* $S_2 = \{L, C, R\}$*. The payoff matrix is as in figure 2.27*

Player 2

Player 1		L	C	R
	T	$\underline{1}, \underline{1}$	$0, \underline{1}$	$0, 0$
	M	$\underline{1}, 0$	$\underline{2}, 1$	$1, \underline{2}$
	B	$0, 0$	$1, \underline{1}$	$\underline{2}, 0$

Figure 2.27: Nash Equilibrium and Weakly Dominated Strategies

In this game (L, T) is the unique Nash equilibrium. However, T is weakly dominated by M and L is weakly dominated by C.

We now consider some more examples.

Variant of Grading Game: Now consider example 2.1.2. There are two Nash equilibria in this game: (α, α) and (β, β) (see figure 2.28). The equilibrium payoff associated with (α, α) is $(0, 0)$ and the equilibrium payoff of (β, β) is $(1, 1)$. Observe we can rank the equilibria: Payoffs of both the players are higher at the equilibrium (α, α), but if they are at the equilibrium (β, β) then none of them has any incentive to deviate unilaterally.

Student 2

Student 1		α	β
	α	$\underline{0}, \underline{0}$	$-1, -1$
	β	$-1, -1$	$\underline{1}, \underline{1}$

Figure 2.28: Variant of Grading Game

Battle of the Sexes: Consider example 2.3. There are two equilibria: (F, F) with a payoff of $(2, 1)$ and (M, M) with a payoff of $(1, 2)$ (see figure 2.29). The woman would prefer the strategy pair (F, F) while the man would rather see (M, M) chosen. However, either one is an equilibrium.

Man

Woman		F	M
	F	$\underline{2}, \underline{1}$	$0, 0$
	M	$0, 0$	$\underline{1}, \underline{2}$

Figure 2.29: Battle of Sexes

Example 2.19. *Consider the Cournot quantity competition. Suppose now there are* 3 *firms.*

We want to find out the Nash equilibrium. Consider the problem of firm 1, it is to

$$\max_{q_1 \geq 0} q_1[a - b(q_1 + q_2 + q_3) - c]$$

The first order condition gives us

$$\text{F.O.C.:} \quad \frac{\partial \Pi}{\partial q_1} = 0 \quad \Rightarrow q_1 = \frac{a-c}{2b} - \frac{q_2 + q_3}{2}.$$

Again we check for the second order condition in order to ensure that this quantity indeed maximizes firm 1's profit.

$$\text{S.O.C.:} \quad \frac{\partial^2 \Pi}{\partial q_1^2} = -2b < 0 \ \text{ (so, S.O.C. is satisfied).}$$

Hence, the best response of firm 1, when it believes that firm 2 produces q_2 and firm 3 produces q_3 is

$$BR_1(q_2, q_3) = \frac{a-c}{2b} - \frac{q_2 + q_3}{2}$$

Similarly, for firm 2 and 3 we have

$$BR_2(q_1, q_3) = \frac{a-c}{2b} - \frac{q_1 + q_3}{2}$$

$$\text{and,} \ \ BR_3(q_1, q_2) = \frac{a-c}{2b} - \frac{q_1 + q_2}{2}$$

The Nash equilibrium satisfies $q_1^* = BR_1(q_2^*, q_3^*) = \frac{a-c}{2b} - \frac{q_2^* + q_3^*}{2}$, $q_2^* = BR_2(q_1^*, q_3^*) = \frac{a-c}{2b} - \frac{q_1^* + q_3^*}{2}$ and $q_3^* = BR_3(q_1^*, q_2^*) = \frac{a-c}{2b} - \frac{q_1^* + q_2^*}{2}$.

From this we get $q_1^* = q_2^* = q_3^* = \frac{a-c}{4b}$, aggregate output is $\frac{3(a-c)}{4b}$ and price $p^* = \frac{a+3c}{4}$.

Exercise 2.7. *Suppose there n firms and they engage in quantity competition. Find out the Nash equilibrium. What happens to equilibrium quantity and price when $n = 1$? What happens to equilibrium quantity, aggregate quantity and price as $n \to \infty$?*

2.5.1 Bertrand Model of Duopoly

We next consider a different model of how duopolists might interact, based on Bertrand's suggestion that firms actually choose prices, rather than quantities as in Cournot's model.

There are two firms in the market. Cost functions are the same: marginal cost is c and there is no fixed cost of production. The market demand function is given by $q = m - np$, where p is the lowest price. Competition takes place as follows: Two firms simultaneously announce their prices p_1 and p_2. Sales for firm 1 are then given by

$$q_1(p_1, p_2) = \begin{cases} m - np_1 & \text{if } p_1 < p_2 \\ \dfrac{1}{2}[m - np_1] & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2 \end{cases}$$

Similarly, sales for firm 2 can be written as

$$q_2(p_1, p_2) = \begin{cases} 0 & \text{if } p_1 < p_2 \\ \dfrac{1}{2}[m - np_2] & \text{if } p_1 = p_2 \\ m - np_2 & \text{if } p_1 > p_2 \end{cases}$$

The firms produce to order so they incur production costs only for an output level equal to their actual sales. Given prices p_1 and p_2, firm 1's profit is therefore equal to $(p_1 - c)q_1(p_1, p_2)$ and firm 2's profit is $(p_2 - c)q_2(p_1, p_2)$. We are interested in finding out Nash equilibrium of this game. For that, first we write down this as a normal-form game.

- There are two players: Firm 1 and Firm 2.
- Each firm's strategy space, in this case which is price, can be represented as $p_i = [0, \infty)$, where $i = \{1, 2\}$.
- We assume that the firm's payoff is simply its profit. Thus, the payoff of firm i is

$$\Pi_i(p_1, p_2) = (p_i - c)q_i(p_1, p_2) \Rightarrow \Pi_i(p_1, p_2) = \begin{cases} (p_i - c)(m - np_i) & \text{if } p_i < p_j \\ (p_i - c)\dfrac{m - np_i}{2} & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j. \end{cases}$$

Let us denote the monopoly price, that is the price which would be charged by a firm where there is no other firm by p^m. So, $p^m = \dfrac{m + nc}{2n}$.
Now observe that the payoff functions are discontinuous, so we cannot differentiate them. However, we will show that there is a unique Nash equilibrium (p_1^*, p_2^*) where both the firms set prices equal to the marginal cost.

In order to do that we will find the firms' best-response functions. But before that let me provide an informal explanation: If firm i charges p_j, it shares the market with firm j; if it charges slightly less, it sells to the entire market. Thus if p_j exceeds c, so that firm i makes a positive profit selling the good at a price slightly below p_j, firm i is definitely better off serving all the market at such a price than serving half of the market at the price p_j. If p_j is very high, however, firm i may be able to do even better: by reducing its price significantly below p_j it may increase its profit, because the extra

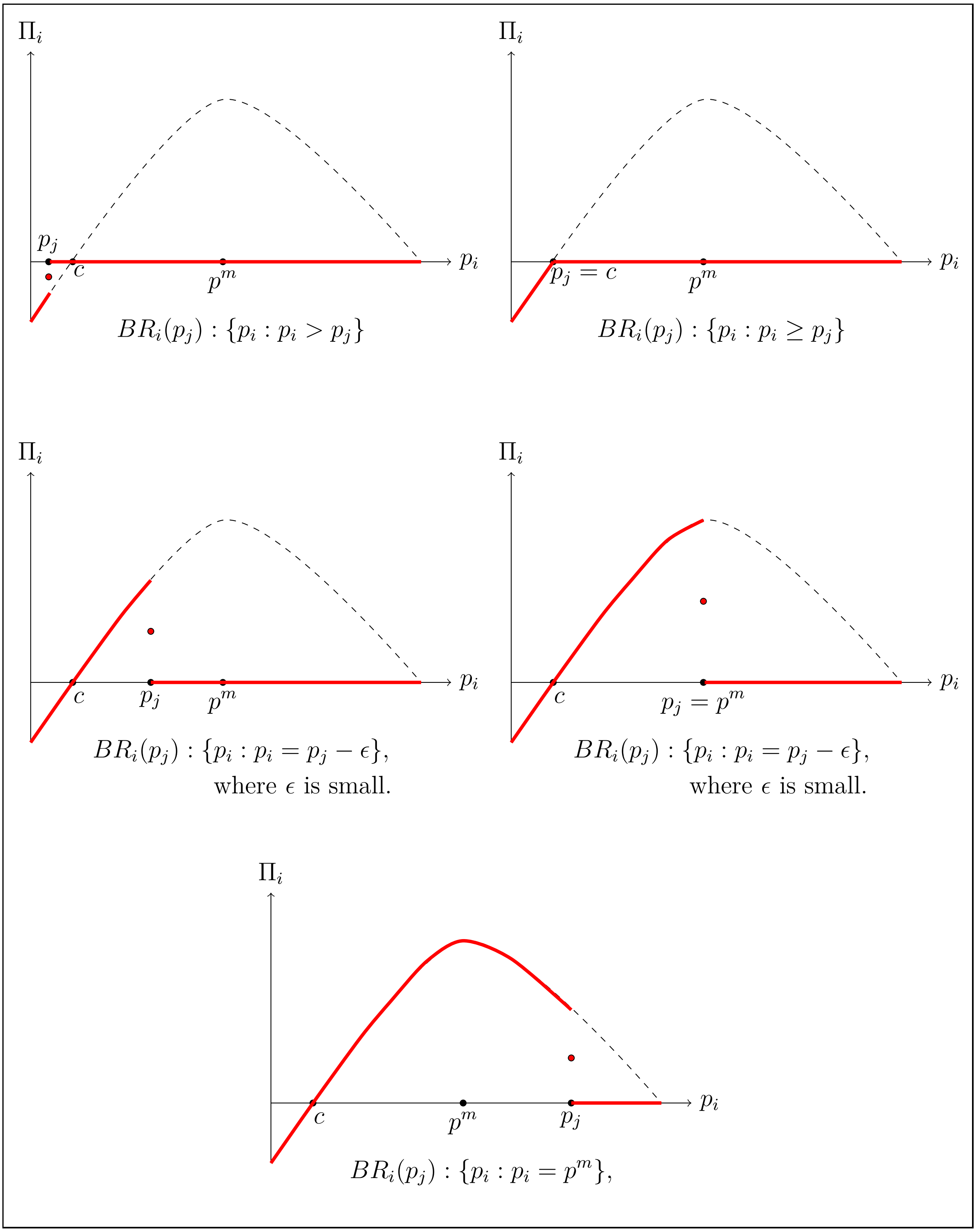


Figure 2.30: Firm i's Payoff and Best Responses in Bertrand duopoly Game

demand engendered by the lower price may more than compensate for the lower revenue per unit sold. Finally, if p_j is less than c, then firm i's profit is negative if it charges a price less than or equal to p_j, whereas this profit is zero if it charges a higher price. Thus in this case firm i would like to charge any price greater than p_j, to make sure that it gets no customers. (Remember that if customers arrive at its door it is obliged to serve them, whether or not it makes a profit by so doing.)
Now we write the best-response function for each firm.

$$BR_i(p_j) = \begin{cases} \{p_i : p_i > p_j\} & \text{if } p_j < c \\ \{p_i : p_i \geq p_j\} & \text{if } p_j = c \\ \{p_j - \epsilon\} & \text{if } c < p_j \leq p^m \quad \text{where } \epsilon \text{ is small} \\ \{p^m\} & \text{if } p^m < p^j. \end{cases}$$

Now we show that *there is a unique Nash equilibrium* (p_1^*, p_2^*) *in the Bertrand duopoly game. In this equilibrium both the firms set prices equal to the marginal cost.*

- We first show that for both the firms setting their prices equal to c is indeed a Nash equilibrium. At these prices both firms earns zero profits. Neither firm can gain by raising its price because it will then make no sales (thereby still earning zero profit); and by lowering it price below c, the sales of the firm would increase but it would incur a loss.

- Next we show that there is no other Nash equilibrium. For this all we need to show is that at any combination of price, at least one of the firms has an incentive to deviate unilaterally.

  - If $p_i < c$ for either $i = 1$ or 2, then the profit of the firm whose price is the lowest (or the profit of both firms, if the prices are the same) is negative, and this firm can increase its profit (to zero) by raising its price to c.
  - If $p_i = c$ and $p_j > c$ then firm i is better off increasing its price slightly, making its profit positive rather than zero.
  - If $p_i > c$ and $p_j > c$, without loss of generality suppose that $p_i \geq p_j$. Then firm i can increase its profit by lowering p_i to slightly below p_j if $q_j(p_1, p_2) > 0$ (that is if $p_j < \frac{m}{n}$) and to p^m if $q_j(p_1, p_2) = 0$ (that is if $p_j \geq \frac{m}{n}$).

2.5.2 Discussion of the Concept Of Nash Equilibrium

It is extremely important to remember that Nash equilibrium assumes **correct** beliefs and best responding with respect to these correct beliefs of other players. Why might it be reasonable to expect players' conjectures about each other's play to be correct? Or in sharper terms, why should we concern ourselves with the concept of Nash equilibrium?

Below we discuss some arguments which have been put forward for the Nash equilibrium concept.

- *Nash equilibrium as a consequence of rational inference.* It is sometimes argued that because each player can think through the strategic considerations faced by his opponents, rationality alone implies that players must be able to correctly forecast what their rivals will play. However, observe that rationality need not lead to players' forecasts to be correct.

- *Nash equilibrium as a necessary condition if there is a unique predicted outcome to a game.* A more satisfying version of the previous idea argues that if there is a unique predicted outcome for a game, then rational players will understand this. Therefore, for no player to wish to deviate, this predicted outcome must be a Nash equilibrium.

- *Nash equilibrium as a self-enforcing agreement.* Another way to express the property of stability is to require that if there is "agreement" to play a particular equilibrium, then, even if the agreement is not binding, it will not be breached: no player will deviate from the equilibrium point, because there is no way to profit from any unilateral violation of the agreement.

- *Nash equilibrium as a stable social convention.* A particular way to play a game might arise over time if the game is played repeatedly and some stable social convention emerges.

- *Equilibrium from the normative perspective.* Consider the concept of equilibrium from the normative perspective of a judge recommending a certain course of action (hopefully based on reasonable and acceptable principles). In that case we should expect the judge's recommendation to be an equilibrium point. Otherwise (since it is a recommendation and not a binding agreement) there will be at least one agent who will be tempted to benefit from not following his end of the recommendation.

Exercise 2.8. *Find out Nash equilibrium (equilibria) of the following games:*

(a) Variant of Prisoner's Dilemma (example 2.2.1).

(b) Voting Game (example 2.4). What happens when there are three candidates?

(c) Games in example 2.5, 2.6, 2.7, 2.9, and 2.14.

(d) Games mentioned in Exercise 2.2

Exercise 2.9. (Guess the average). *Fifty people are playing the following game. Each player writes down, on a separate slip of paper, one integer in the set* $\{0, 1, ..., 100\}$*, alongside his name. The game master then reads the numbers on each slip of paper, and*

calculates the average x of all the numbers written by the players. The winner of the game is the player (or players) who wrote down the number that is closest to $\frac{2}{3}$ of x. The winners equally divide the prize of $\$1,000$ *between them.*
Describe this as a strategic-form game, and find all the Nash equilibria of the game.

Exercise 2.10. (Stag Hunt) *Each of a group of hunters has two options: she may remain attentive to the pursuit of a stag, or catch a hare. If all hunters pursue the stag, they catch it and share it equally; if any hunter devotes her energy to catching a hare, the stag escapes, and the hare belongs to the defecting hunter alone. Each hunter prefers a share of the stag to a hare.*
Describe this as a strategic-form game, and find all the Nash equilibria of the game.

Let us consider some more examples.

Example 2.20. (Another Variant of Grading Game) *Suppose all the students in a class are asked to choose between α and β simultaneously. If a student chooses "β" he/she immediately gets B^+. If the student chooses "α", then his/her payoff depends on the choices made by the other students. If at least* 90% *students choose α, then all of those who chose α get A, otherwise they get F.*

Let us represent this as a normal form game.

- Players: Students (suppose there are N students).
- Strategies: $s_i = \{\alpha, \beta\}\ \forall i \in \{1, ..., N\}$.
- Payoff of each student is

$$u_i(\beta, s_{-i}) = B^+$$
$$u_i(\alpha, s_{-i}) = \begin{cases} A & \text{if 90\% students choose } \alpha \\ F & \text{otherwise.} \end{cases}$$

In this game there are two Nash equilibria – everyone chooses α: $(\alpha, ..., \alpha)$ and everyone chooses β: $(\beta, ...\beta)$. Here the Nash equilibrium $(\alpha, ..., \alpha)$ may seem more likely to attract players' attentions than others. To use the terminology of Schelling (1960), $(\alpha, ..., \alpha)$ may seem to be *focal*. Similarly, in example 2.1.2 choosing (β, β) or in example 2.13 (F, F) seem more likely to be focal.

However, whether players in coordination games can find a focal point depends on their having some commonly known point of contact, whether historical, cultural, or linguistic. Without genuine convergence of expectations about actions, they may choose the worse equilibrium or, worse still, players may fail to coordinate and get zero. Communication and many social institutions serve to coordinate beliefs and behaviour. They create durable systems of beliefs in which economic agents can have confidence. They align our expectations and give us the security of knowing that what we expect will actually take place. The theory of Nash equilibrium, though, is neutral about the equilibrium that will occur in a game with many equilibria.

Example 2.21. (Bank Runs) *Two investors have each deposited D with a bank. The bank has invested in a long-term project. If the bank is forced to liquidate its investment before the project matures, a total of $2r$ can be recovered, where $D > r$ (hence, $r > 2r - D$). If the bank allows the investment to reach maturity, however, the project will pay out a total of $2R$, where $R > D$.*

Let us represent this as a normal form game.

- Players: Investor 1 and Investor 2.
- Strategies: $S_i = \{\text{withdraw, don't withdraw}\}$
- Payoff is given in the payoff matrix in Figure 2.31.

		Investor 2	
		Withdraw	Don't withdraw
Investor 1	Withdraw	$\underline{r}, \underline{r}$	$D, 2r - D$
	Don't withdraw	$2r - D, D$	$\underline{R}, \underline{R}$

Figure 2.31: Bank Runs

There are two Nash equilibria (Withdraw,Withdraw) and (Don't withdraw,Don't withdraw) and the corresponding payoffs are (r, r) and (R, R). The first of these two outcomes can be interpreted as a run on bank. If Investor 1 believes that Investor 2 will withdraw then Investor 1's best response is to withdraw as well, even though both investors would be better off if they had not. This is also one type of coordination problem.

Observe, these games are different from Prisoner's Dilemma (example 2.2, as discussed before, in that game communication does not help in achieving the better outcome (C, C) as each player has an incentive to deviate unilaterally.

Example 2.22. (Partnership Game) *Two people come together to open a partnership firm. The firm's profit, which the partners share, depends on the effort that each person expends on the job. Suppose that the profit is $\pi = 4(e_1 + e_2 + \nu e_1 e_2)$, where ν measures how complementary the tasks of the partners are and we assume $\nu \in [0, \frac{1}{4}]$. e_i is the amount of effort expended by partner i and we assume that $e_i \in [0, 4]$. Effort is costly and let that cost be e_i^2 in monetary terms.*

We also assume that it is not possible to write a contract and that each partner chooses effort independently, and simultaneously. The partners seek to maximize their individual share of the firm's profit net of the cost of effort.

Let us write down this as a normal form game:

- Two players: Player 1 (Partner 1) and Player 2 (Partner 2).
- Each partner's strategy is to choose effort $e_i = [0, 4]$
- Payoff of Player i is $u_i = 2(e_i + e_j + \nu e_i e_j) - e_i^2$, where $i, j \in \{1, 2\}$ and $i \neq j$.

Now consider Partner i's problem, given Partner j's effort, it is to

$$\max_{e_i} 2(e_i + e_j + \nu e_i e_j) - e_i^2$$

$$F.O.C. : \frac{\partial u_i}{\partial e_i} = 0 \quad \Rightarrow \; 2 + 2\nu e_j - 2e_i = 0 \quad \Rightarrow \; e_i = 1 + \nu e_j.$$

Again we check the second order condition in order to ensure that this effort level indeed maximizes Partner i's utility

$$S.O.C. : \frac{\partial^2 u_i}{\partial e_i^2} = -2 < 0 \; \text{(hence, S.O.C. is satisfied)}.$$

We draw both the players' best responses in Figure 2.32 (for $\nu = \frac{1}{2}$).

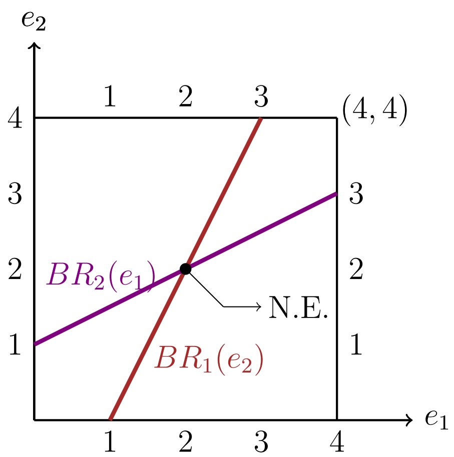


Figure 2.32: Partnership Game

The unique Nash equilibrium of this game is $\left(\frac{1}{1-\nu}, \frac{1}{1-\nu}\right)$ (when $\nu = \frac{1}{2}$ it is $(2, 2)$) and the corresponding payoffs are $\left(\frac{3-2\nu}{(1-\nu)^2}, \frac{3-2\nu}{(1-\nu)^2}\right)$ (when $\nu = \frac{1}{2}$ the payoffs are $(8, 8)$).

Exercise 2.11. *Find out the never-best responses of both the players and show two rounds of elimination (the way we have done in Cournot).*

Observe, in this game, increases in effort by one of the partners is beneficial to him, up to a point. Furthermore, increasing a partner's effort is more valuable the greater is the other partner's effort level. To see this, note that the first derivative of player 1's payoff function with respect to his own effort, e_1, increases as e_2 increases. The same is true for player 2's payoff. The relation between the partners' strategies is *complementary* here. Formally,

Definition 2.10. *In games with strategic complements best response of each player is weakly increasing in actions of the other players. Alternatively, the cross partial derivative of player i's payoff function with respect to player i's strategy and any other player's strategy is non-negative*

$$\text{for each } i, j \in N \text{ and } i \neq j \ \ \frac{\partial^2 u_i(s)}{\partial s_i \partial s_j} \geq 0 \ \ \forall s \in S.$$

In the partnership game, because of the strategic complementarity, each player's best-response function is increasing in the mean belief about the partner's strategy. That is, as e_j increases, so does the optimal e_i in response. In terms of the graph in Figure 2.32, the players' best-response functions are *positively sloped* and as ν increases the slope of each best-response function also increases.

Conversely, in the games with strategic substitutes the best response of each player is decreasing in the actions of the other players. Formally,

Definition 2.11. *In a game with strategic substitutes, the cross partial derivative of player i's payoff function with respect to player i's strategy and any other player's strategy is negative*

$$\text{for each } i, j \in N \text{ and } i \neq j \ \ \frac{\partial^2 u_i(s)}{\partial s_i \partial s_j} < 0 \ \ \forall s \in S.$$

Observe, in Cournot duopoly model quantities are strategic substitutes.

Exercise 2.12. *Consider the voting game (example 2.4).*

(a) Find out all the Nash equilibria.

(b) Find out all the Nash equilibria when there are three candidates.

Example 2.23. (Strategic Voting) *Suppose there are three voters (Voter 1, Voter 2 and Voter 3) and three candidates (A, B, and C). The candidate which gets the maximum votes win, if all the candidates get same number of votes, then candidate A wins. Voters' preferences are given in Figure 2.33.*

Suppose, a voter's utility when his most preferred candidate wins be 2*, when his second most preferred candidate wins be* 1 *and when his least preferred candidate wins be* 0*.*

We represent this in a normal form game

Voter 1	Voter 2	Voter 2
A	B	C
B	A	B
C	C	A

Figure 2.33: Strategic Voting

- Players: {Voter 1, Voter 2 and Voter 3}.
- Strategies: $S_i = \{A, B, C\}$, where $i = \{1, 2, 3\}$.
- Payoffs are as follows

$$u_1 = \begin{cases} 2 & \text{if } A \text{ wins} \\ 1 & \text{if } B \text{ wins} \\ 0 & \text{if } C \text{ wins} \end{cases} \quad u_2 = \begin{cases} 2 & \text{if } B \text{ wins} \\ 1 & \text{if } A \text{ wins} \\ 0 & \text{if } C \text{ wins} \end{cases} \quad u_3 = \begin{cases} 2 & \text{if } C \text{ wins} \\ 1 & \text{if } B \text{ wins} \\ 0 & \text{if } A \text{ wins} \end{cases}$$

If all the voters vote sincerely (that is vote for his most preferred candidate) then candidate A will win. In this case payoffs are $(2, 1, 0)$. However, observe this is not a Nash equilibrium as Voter 3 has a profitable deviation – if he deviates and vote for candidate B, then candidate B will win and his utility will be 1. In this game the unique Nash equilibrium is (A, B, B) (check!).
Now we consider an example where there are three players.

Example 2.24. (Three Players) *Suppose three roommates; Anu, Neha and Puja; in a hostel room are deciding whether to contribute towards the creation of a flower garden in their small balcony. If all of them contribute then that will produce the largest and best garden, if two of them contribute then that will produce a medium garden and if only one of them contributes then a small garden will be produced.*

Suppose, each of them is happy to have the garden and happier as its size and splendor increase, but each is reluctant to contribute because of the cost that she must incur to do so. Let us assume some numbers which conform with this: Let cost of contribution be 3, utility from the largest garden (where all of them contributes) be 8, the utility from medium sized (where two of them contribute) garden be 6 and the utility from the smallest garden be 4, finally utility from no garden be 2.

We represent this in a normal-form game first.

- Players: Anu (Player 1), Neha (Player 2) and Puja (Player 3).
- Strategies: "Contribute" (C) and "Don't contribute" (D). $S_i = \{C, D\}$, where $i = \{1, 2, 3\}$.

- Payoffs: Given in Figure 2.34. Player 1 chooses a row, Player 2 chooses a column, and Player 3 chooses a matrix.

	C	D
C	$5,5,5$	$3,6,3$
D	$6,3,3$	$4,4,1$

C

	C	D
C	$3,3,6$	$1,4,4$
D	$4,1,4$	$2,2,2$

D

Figure 2.34: Three-Player Game

Here the first component of each box corresponds to Player 1's payoff, second component corresponds to Player 2's payoff and the third component corresponds to Player 3's payoff. Let us first check whether there is any dominant strategy for any player.
Comparing Player 1's strategies C and D we find that

- If Player 2 plays C and Player 3 player C, the payoff to Player 1 under strategy C is 5, compared to 6 under strategy D.
- If Player 2 plays D and Player 3 player C, the payoff to Player 1 under strategy C is 3, compared to 4 under strategy D.
- If Player 2 plays C and Player 3 player D, the payoff to Player 1 under strategy C is 3, compared to 4 under strategy D.
- If Player 2 plays D and Player 3 player D, the payoff to Player 1 under strategy C is 1, compared to 2 under strategy D.

So, we see that independently of whichever strategies are played by Player 2 and Player 3, strategy D always yields a higher payoff to Player 1 than strategy C. So strategy D strictly dominates strategy C or in other words, strategy C is strictly dominated by strategy strategy D.

Similarly check that for Player 2 and Player 3 also, strategy C is strictly dominated by strategy D. So, the unique solution of this game is (D, D, D) and the corresponding payoff is $(2, 2, 2)$.

Given Claim 2.1, we know that (D, D, D) is the unique Nash equilibrium, but let us find that out using best-response as well. The best responses of each player are underlined in Figure 2.35 and observe (D, D, D) is indeed a Nash equilibrium.

Next we introduce a concept called **strict Nash equilibrium**. For this consider again the definition of Nash equilibrium, carefully observe that we say that at Nash equilibrium no player has any *profitable* unilateral deviation. That is, no player can do anything *better* by deviating, but the players may have deviation in which, given the

	C	D
C	$5, 5, 5$	$3, \underline{6}, 3$
D	$\underline{6}, 3, 3$	$\underline{4}, \underline{4}, 1$

C

	C	D
C	$3, 3, \underline{6}$	$1, \underline{4}, \underline{4}$
D	$\underline{4}, 1, \underline{4}$	$\underline{2}, \underline{2}, \underline{2}$

D

Figure 2.35: Three-Player Game: Best-Response Analysis

	L	C	R
T	$\underline{1}, \underline{1}$	$\underline{1}, 0$	$0, 0$
B	$0, 0$	$0, \underline{1}$	$\underline{1}, 0$

Game 1. A game with a unique Nash equilibrium, which is a strict equilibrium.

	L	C	R
T	$\underline{1}, \underline{1}$	$\underline{1}, 0$	$0, \underline{1}$
B	$\underline{1}, 0$	$0, \underline{1}$	$\underline{1}, 0$

Game 2. A game with a unique Nash equilibrium, which is not a strict equilibrium.

Figure 2.36: Strict and Not Strict Nash Equilibrium

other players' strategies, he is equally well-off. To understand this consider the following two examples in figure 2.36.

In Game 1, (T, L) is the unique Nash equilibrium. If Player 1 deviates to strategy B, while Player 2 is playing L, he will be *strictly* worse off. Similarly, if Player 2 deviates to strategy C or R, while Player 1 plays T, then he will be strictly worse off. We call such a Nash equilibrium **strict Nash equilibrium**. Formally,

Definition 2.12. *A Nash equilibrium s^* is termed strict if every deviation undertaken by a player yields a definite loss for that player, i.e.,*

$$u_i(s^*) > u_i(s_i, s^*_{-i}) \quad \text{for each player } i \in N \ \text{ and each strategy} s_i \in S_i \setminus \{s_i^*\}.$$

In other words, player i's strategy s_i^* is the only best response to s^*_{-i}: $\{s_i^*\} = BR_i(s^*_{-i})$ for each player $i \in N$.

In Game 2 also, (T, L) is the unique Nash equilibrium. However, now if Player 1 deviates to strategy B, while Player 2 plays L, he neither gains nor loses, that is he does not become strictly worse off from this deviation. Similarly, Player 2 also does not become strictly worse if he deviates to R.

Exercise 2.13. *Consider the games which we have analyzed so far. Check which Nash equilibria among them are strict.*

Another important terminology is *symmetric Games*. A game is symmetric if each player has exactly the same strategy set and the payoff functions are identical. A game such as Cournot is called a symmetric game. Roughly speaking, a symmetric game is one in which each player is equal to every other player: each has the same opportunities, and the same actions yield the same payoffs. Equivalently, you can think of a symmetric game as one in which the players' names are irrelevant and only their actions are relevant.

Exercise 2.14. *Consider again the games which we have analyzed so far. Point out the games which are symmetric.*

Consider example 2.10, we can represent it using the following payoff matrix

		Player 2	
		Heads	Tails
Player 1	Heads	$-1, 1$	$1, -1$
	Tails	$1, -1$	$-1, 1$

Figure 2.37: Matching Pennies

Observe, in this game no pair of strategies is Nash equilibrium, since if the players' strategies match – (Heads, Heads) or (Tails, Tails) – then Player 1 prefers to switch strategies, while if the strategies do not match – (Heads, Tails) or (Tails, Heads) – then Player 2 prefers to do so. The distinguishing feature of Matching Pennies is that each player would like to outguess the other. We now introduce the notion of *mixed strategy*. We will then extend the definition of Nash equilibrium to include mixed strategies. We shall see in this extended framework, Matching Pennies game has a Nash equilibrium. In fact, Nash showed that "the mixed extension of every finite game has a Nash equilibrium" (proof is beyond the scope of this course).

Relevant Parts of the Reference Book: Chapter 8,9,10.

2.6 Mixed Strategy

When players choose to act unsystematically, they pick from among their pure strategies in some random way. Formally, a mixed strategy for Player i is a probability distribution over (some or all of) the strategies in S_i. We will hereafter refer to the strategies in S_i as Player i's *pure strategies*. In Matching Pennies, for example, S_i consists of the two pure strategies Heads and Tails, so a mixed strategy for Player i is the probability distribution $(p, 1-p)$, where p is the probability of playing Heads (H) and $1-p$ is the probability of playing Tails (T), and $0 \leq q \leq 1$. The mixed strategy $(1, 0)$ is simply the pure strategy H; likewise, the mixed strategy $(0, 1)$ is the pure strategy T.

As a second example of a mixed strategy, recall example 2.17, where Player 1 has the pure strategies Top (T), Middle (M) and Bottom (B). Here a mixed strategy for Player

1 is the probability distribution $(p, q, 1-p-q)$, where p is the probability of playing Top, q is the probability of playing Middle, and $1-p-q$ is the probability of playing Bottom. As before, $0 \leq p \leq 1$, and now $0 \leq q \leq 1$ and $0 \leq p+q \leq 1$. In this game, the mixed strategy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ puts equal probability on T, M and B whereas $(\frac{1}{2}, \frac{1}{2}, 0)$ puts equal probability on T and M but no probability on B. As always, a player's pure strategies are simply the limiting cases of the player's mixed strategies – here Player 1's pure strategy T is the mixed strategy $(1, 0, 0)$, for example.

More generally, suppose that Player i has K pure strategies: $S_i \equiv (s_i^1, ..., s_i^K)$. Then a mixed strategy for player i is a probability distribution $\sigma_i = \left(\sigma_i(s_i^1), ..., \sigma_i(s_i^K)\right)$, where $\sigma_i(s_i^k)$ is the probability that player i will play strategy s_i^k, for $k = 1, ..., K$. Since, $\sigma_i(s_i^k)$ is a probability, we require $0 \leq \sigma_i(s_i^k) \leq 1$ for $k = 1, ..., K$ and $\sum_{k=1}^{K} \sigma_i(s_i^k) = 1$. We will use σ_i to denote an arbitrary mixed strategy from the set of probability distributions over S_i, just as we use s_i to denote pure strategy from S_i.

Definition 2.13. *In the normal-form game* $\mathcal{G} \equiv \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$*, suppose* $S_i \equiv (s_i^1, ..., s_i^K)$*. Then a* **mixed strategy** *for Player i is a probability distribution* $\sigma_i = \left(\sigma_i(s_i^1), ..., \sigma_i(s_i^K)\right)$*, where* $0 \leq \sigma_i(s_i^k) \leq 1$ *for* $k = 1, ..., K$ *and* $\sum_{k=1}^{K} \sigma_i(s_i^k) = 1$*.*

Under mixed strategy, players are assumed to randomize independently, i.e., how a player randomizes does not depend on how others randomize. Consider again the game Matching Pennies. Suppose Player 1 plays the mixed strategy H with probability $\frac{3}{4}$ and Tails with probability $\frac{1}{4}$. Suppose Player 2 plays H with probability $\frac{1}{4}$ and T with probability $\frac{3}{4}$. Then, the mixed strategy profile is, in general terms,

$$\sigma \equiv (\sigma_1, \sigma_2) = \left((\sigma_1(H), \sigma_1(T)), (\sigma_2(H), \sigma_2(T))\right) = \left((\frac{3}{4}, \frac{1}{4})(\frac{1}{4}, \frac{3}{4})\right).$$

From this, the probability with which each pure strategy profile is played can be computed (using independence). These probabilities are shown in figure 2.38. A player

		Player 2 H	**Player 2** T
Player 1	H	$\frac{3}{16}$	$\frac{9}{16}$
	T	$\frac{1}{16}$	$\frac{3}{16}$

Figure 2.38: Mixed strategies - probability of all pure strategy profiles

computes the utility from a mixed strategy profile using expected utility. The mixed strategy profile σ gives players payoffs as follows:

$$\begin{aligned}
EU_1(\sigma) &= \sigma_1(H)\cdot\sigma_2(H)\cdot(-1)+\sigma_1(H)\cdot\sigma_2(T)\cdot(1)+\sigma_1(T)\cdot\sigma_2(H)\cdot(1)+\sigma_1(T)\cdot\sigma_2(H)\cdot(-1)\\
&= \frac{3}{16}\cdot(-1)+\frac{9}{16}\cdot(1)+\frac{1}{16}\cdot(1)+\frac{3}{16}\cdot(-1)\\
&= \frac{1}{4}.\\
EU_2(\sigma) &= \sigma_1(H)\cdot\sigma_2(H)\cdot(1)+\sigma_1(H)\cdot\sigma_2(T)\cdot(-1)+\sigma_1(T)\cdot\sigma_2(H)\cdot(-1)+\sigma_1(T)\cdot\sigma_2(H)\cdot(1)\\
&= \frac{3}{16}\cdot(1)+\frac{9}{16}\cdot(-1)+\frac{1}{16}\cdot(-1)+\frac{3}{16}\cdot(1)\\
&= -\frac{1}{4}.
\end{aligned}$$

Let us denote the *mixed extension* of the game $\mathcal{G}$[9] by $\Delta\mathcal{G}$, that is

$$\mathcal{G} \equiv \langle N, \{\Delta S_i\}_{i\in N}, \{EU_i\}_{i\in N}\rangle$$

Nash showed that *"the mixed extension of every finite game has a Nash equilibrium"* (we will not prove this theorem, in this course). Though mixed strategies guarantee existence of Nash equilibrium in finite games. But, it is not clear why a player will randomize in the precise way prescribed by a mixed strategy Nash equilibrium, specially given the fact he is indifferent between the pure strategies in the support of such a Nash equilibrium. There are no clear answers to this question. However, following are some arguments to validate that mixed strategies can be part of Nash equilibrium play.

- Players randomize deliberately. For instance, in zero-sum games with two players, players may randomize. In games like Poker, players have been shown to randomize.
- Mixed strategy equilibrium can be thought to be a belief system - if σ^* is a Nash equilibrium, then σ_i^* describes the belief that opponents of Player i have on Player i's. This means that Player i may not actually randomize but his opponents collectively believe that σ_i^* is the strategy he will play. Hence, a mixed strategy equilibrium is just a steady state of beliefs.
- One can think of a strategic form game being played over time repeatedly (payoffs and actions across periods do not interact). Suppose players choose a best response in each period assuming time average of plays of past (with some initial conditions on how to choose strategies). In particular, they observe that opponents have been playing a strategy A for $\frac{3}{4}$ times and another strategy B for the remaining time. So, they optimally respond by forming this as their beliefs. It has been shown that such plays eventually converge to a steady state where the average play of each player is some mixed strategy in some class of games.

[9]Recall definition 2.1

- Another interpretation that is provided by Nash himself interprets Nash equilibrium as population play. There are two pools of large population. We draw a player at random from each pool and pair them against each other. The strategy of that player will reflect the expected strategy played by the population and will represent a mixed strategy. So, Nash equilibrium represents some kind of stationary distribution of pure strategies in such population.

Now that we have extended the strategy space, the question we ask is what happens to dominated and dominant strategies, and Nash equilibria. We will also study relationship between undominated strategies and best-responses and a new concept called *rationalizability*.

2.6.1 Domination

In the following example we show that a pure strategy that is not dominated by any pure strategy may be dominated by a mixed strategy.

Example 2.25. *Player 1 has three strategies $S_1 = \{T, M, B\}$ and Player 2 has two strategies $S_2 = \{L.R\}$. Payoffs are given in Figure 2.39*

Player 2

Player 1		L	R
	T	$3,1$	$0,4$
	M	$0,2$	$3,1$
	B	$1,0$	$1,2$

Figure 2.39: Mixed strategies may dominate pure strategies

In this game, strategy B is not dominated by any pure strategy for Player 1. However, the mixed strategy $\frac{1}{2}T$ and $\frac{1}{2}M$ strictly dominates the pure strategy B. Hence, B is a strictly dominated strategy for Player 1 in the mixed extension of the game.
Hence, we redefine strictly dominated strategy incorporating mixed strategies.

Definition 2.14. *A pure strategy s_i of Player i is* **strictly dominated** *if there is a strategy (pure or mixed) $\sigma_i \in \Delta S_i$ of Player i such that for each strategy vector $s_{-i} \in S_{-i}$*

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}).$$

Exercise 2.15. *Consider the following games and find out strictly dominated strategies.*

(a) The game in example 2.16

(b) Player 1 has three strategies $S_1 = \{T, M, B\}$ and Player 2 has two strategies $S_2 = \{L.R\}$. Payoffs are given in Figure 2.40

		Player 2	
		L	R
	T	$4,1$	$0,2$
Player 1	M	$0,0$	$4,0$
	B	$1,3$	$1,2$

Figure 2.40: Mixed strategies may dominate pure strategies

In the next example we show that even if a group of pure strategies are not strictly dominated, a mixed strategy with only these strategies in its support may be strictly dominated.

Example 2.26. *Player 1 has three strategies* $S_1 = \{T, M, B\}$ *and Player 2 has two strategies* $S_2 = \{L.R\}$. *Payoffs are given in Figure 2.41*

		Player 2	
		L	R
	T	$3,1$	$0,4$
Player 1	M	$0,2$	$3,1$
	B	$2,0$	$2,2$

Figure 2.41: Mixed strategies may be dominated

In this game, the pure strategies T and M are not strictly dominated. But the mixed strategy $\frac{1}{2}T + \frac{1}{2}M$ is strictly dominated by pure strategy B.

2.6.2 Strict Dominance and Best Response

There is a precise relation between strict dominance and best response. For a given game, let UD_i be the set of strategies for player i that are *not* strictly dominated (undominated). Let B_i be the set of strategies for player i that are best responses, over all of the possible beliefs of player i. Mathematically,

$$B_i = \{s_i |\text{ there is a belief } \theta_i \in \Delta S_{-i} \text{ such that } s_i \in BR_i(\theta_i)\}.$$

That is, if a strategy s_i is a best response to *some* possible belief of player i, then s_i is contained in B_i.

Let us consider the following example to understand the relationship between UD_i and B_i.

Example 2.27. *Player 1 again has three strategies* $S_1 = \{T, M, B\}$ *and Player 2 has two strategies* $S_2 = \{L, R\}$ *Payoffs are given in the following payoff matrix (figure 2.42).*

Player 2

		L	R
Player 1	T	$6,3$	$0,1$
	M	$2,1$	$5,0$
	B	$3,2$	$3,1$

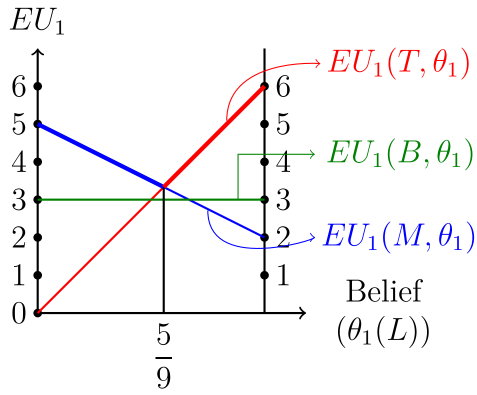


Figure 2.42: Strict Dominance and Best Response

In this game, for Player 2 R is strictly dominated by L, so $UD_2 = \{L\}$. And, R is never a best response or in other words, L is the best response to all possible beliefs. Therefore, $B_2 = \{L\}$, which implies $B_2 = UD_2$.

Now consider Player 1. Observe, strategy B is strictly dominated by the mixed strategy $\frac{1}{3}T + \frac{2}{3}M$. Hence, $UD_1 = \{T, M\}$. And, Player 1's best-response for different beliefs are

$$BR_1(\theta_1) = \begin{cases} \{M\} & \text{when } \theta_1(L) \in [0, \frac{5}{9}) \\ \\ \{M, T\} & \text{when } \theta_1(L) = \frac{5}{9} \\ \\ \{T\} & \text{when } \theta_1(L) \in (\frac{5}{9}, 1]. \end{cases}$$

Hence, $B_1 = UD_1$.
This result is true in general, for two players: *For a two player game $B_1 = UD_1$ and $B_2 = UD_2$.*
Alternatively, we define *never a best response* in this mixed extension of the game:

Definition 2.15. *A strategy $\sigma_i \in \Delta S_i$ is* **never a best response** *for Player i if for every $\sigma_{-i} \in \Delta S_{-i}$,*

$$\sigma_i \notin BR_i(\sigma_{-i}).$$

The following claim is a straightforward observation.

Claim 2.2. *If a strategy is strictly dominated, then it is never a best response.*

The connection between never best response strategies and strictly dominated strategies is deeper. Indeed, in two-player games, a pure strategy is strictly dominated if and only if it is never a best response.

2.6.3 Rationalizability

What strategy choices in games can be justified on the basis of rationality alone? The strategies that remain after the iterative deletion are the strategies that a rational player can justify, or *rationalize*, affirmatively with some reasonable conjecture about the choices of his rivals; that is, with a conjecture that does not assume that any player will play a strategy that is never a best response or one that is only a best response to a conjecture that someone else will play such a strategy, and so on. As a result, the set of strategies surviving this iterative deletion process can be said to be precisely the set of strategies that can be played by rational players in a game in which the players' rationality and the structure of the game are common knowledge. They are known as *rationalizable strategies*.

Definition 2.16. *The strategies in ΔS_i that survive the iterative elimination of strategies that are never a best response are known as player i's* **rationalizable strategies**.

Note that the set of rationalizable strategies can be no larger than the set of strategies surviving iterative removal of strictly dominated strategies because at each stage of the iterative process, all the strategies that are strictly dominated at that stage are eliminated. Let us consider an example.

Example 2.28. *Player 1 has four strategies: $S_1 = \{a_1, a_2, a_3, a_4\}$ and Player 2 has four strategies: $S_2 = \{b_1, b_2, b_3, b_4\}$. The payoff are given in Figure 2.43*

		Player 2			
		b_1	b_2	b_3	b_4
Player 1	a_1	$0, 7$	$2, 5$	$7, 0$	$0, 1$
	a_2	$5, 2$	$3, 3$	$5, 2$	$0, 1$
	a_3	$7, 0$	$2, 5$	$0, 7$	$0, 1$
	a_4	$0, 0$	$0, -2$	$0, 0$	$10, -1$

Figure 2.43: Rationalizable Strategies

In the first round of elimination we can eliminate strategy b_4 which is strictly dominated by the mixed strategy $\frac{1}{2}b_1 + \frac{1}{2}b_3$. Once the strategy b_4 is eliminated, strategy a_4 can be eliminated because it is strictly dominated by a_2 once b_4 is eliminated. At this point no further strategies can be ruled out: a_1 is a best response to b_3, a_2 is a best response to b_2, and a_3 is a best response to b_1. Similarly, b_1, b_2 and b_3 are best responses to a_1, a_2 and a_3 respectively. Thus, the set of rationalizable pure strategies for Player 1 is $\{a_1, a_2, a_3\}$ and the set $\{b_1, b_2, b_3\}$ is rationalizable for Player 2.

Note for each of these rationalizable strategies, a player can construct a *chain of justification* for his choice that never relies on any player believing that another player will play a strategy that is never a best response. In the above example (example 2.28), Player 1 can justify choosing a_2 by the belief that Player 2 will play b_2 which Player 1 can justify to himself by believing that Player 2 will think that he is going to play a_2, which is reasonable if Player 1 believes that Player 2 is thinking that he, Player 1, thinks that Player 2 will play b_2 and so on. Thus Player 1 can construct an (infinite) chain of justification for playing strategy a_2, $(a_2, b_2, a_2, b_2, ...)$, where each element is justified using the next element in the sequence.

Similarly, Player 1 can rationalize playing strategy a_1 with the chain of justification $(a_1, b_3, a_3, b_1, a_1, b_3, a_3, b_1, a_1, ...)$. Here Player 1 justifies playing a_1 by believing that Player 2 will play b_3. He justifies the belief that Player 2 will play b_3 by thinking that Player 2 believes that he, Player 1, will play a_3. He justifies this belief by thinking that Player 2 thinks that he, Player 1, believes that Player 2 will play b_1, and so on.

Suppose, however, that Player 1 tried to justify a_4. He could do so only by a belief that Player 2 would play b_4, but there is *no* belief that Player 2 could have that would justify b_4. Hence, Player 1 cannot justify playing the nonrationalizable strategy a_4.

It can be shown that under fairly weak consitions a player always has at least one rationalizable strategy. However, players may have many rationalizable strategies as in example 2.43.

Next we will learn how to compute Nash equilibrium in this extended strategy space. Then we will study the relationship between rationalizable strategy and Nash equilibrium.

2.6.4 Computing Nash Equilibrium

We start with an important lemma – *if a mixed strategy is a best response then each of the pure strategies involved in the mix must itself be a best response. In particular, each must yield the* **same** *expected payoff.*

We state this formally below and provide a proof. After the proof we will discuss why this result is important, in particular we will discuss how this result can help us in finding Nash equilibrium in the mixed extension of games.

Lemma 2.1. (Indifference Principle) *If Player i's mixed strategy σ_i is a best response to the (mixed) strategies of the other players, σ_{-i}, then for each pure strategy s_i such that $\sigma_i(s_i) > 0$, it must be the case that s_i is itself a best response to σ_{-i}. Formally, suppose $\sigma_i \in BR_i(\sigma_{-i})$ and $\sigma_i(s_i) > 0$, then $s_i \in BR_i(\sigma_{-i})$.*
In particular, $EU_i(s_i, \sigma_{-i})$ must be the same for all such strategies.

We prove this lemma formally first and then discuss the intuition.
Proof. Suppose, $\sigma_i \in BR_i(\sigma_{-i})$. Let $S_i(\sigma_i) := \{s_i \in S_i : \sigma_(s_i) > 0\}$. If $\left|S_i(\sigma_i)\right| = 1$, then the claim is obviously true. Else pick $\hat{s}_i, \tilde{s}_i \in S_i(\sigma_i)$. We argue that $EU_i(\hat{s}_i, \sigma_{-i}) =$

$EU_i(\tilde{s}_i, \sigma_{-i})$. Suppose not and $EU_i(\hat{s}_i, \sigma_{-i}) > EU_i(\tilde{s}_i, \sigma_{-i})$. Then,

$$\begin{aligned}
EU_i(\sigma_i, \sigma_{-i}) &= \sum_{s_i \in S_i(\sigma_i)} EU_i(s_i, \sigma_{-i})\sigma_i(s_i) \\
&= EU_i(\hat{s}_i, \sigma_{-i})\sigma_i(\hat{s}_i) + EU_i(\tilde{s}_i, \sigma_{-i})\sigma_i(\tilde{s}_i) + \sum_{s_i \in S_i(\sigma_i)\setminus\{\hat{s}_i, \tilde{s}_i\}} EU_i(s_i, \sigma_{-i})\sigma_i(s_i) \\
&< EU_i(\hat{s}_i, \sigma_{-i})[\sigma_i(\hat{s}_i) + \sigma_i(\tilde{s}_i)] + \sum_{s_i \in S_i(\sigma_i)\setminus\{\hat{s}_i, \tilde{s}_i\}} EU_i(s_i, \sigma_{-i})\sigma_i(s_i) \\
&= EU_i(\sigma_i^{'}, \sigma_{-i})
\end{aligned}$$

where $\sigma_i^{'}$ is the new mixed strategy of Player i, where he plays $\hat{s}_i$ with probability $\sigma_i(\hat{s}_i) + \sigma_i(\tilde{s}_i)$ and $\tilde{s}_i$ with probability zero, and every other strategy s_i in $S_i(\sigma_i)$ is played with probability $\sigma_i(s_i)$. But this contradicts the fact that $\sigma_i \in BR_i(\sigma_{-i})$. This means that, $EU_i(\hat{s}_i, \sigma_{-i}) = EU_i(\tilde{s}_i, \sigma_{-i})$ for all $\hat{s}_i, \tilde{s}_i \in S_i(\sigma_i)$. This proves the lemma. ■

The intuition is as follows: Suppose it *were* not true. Then there must be at least one pure strategy s_i that is assigned positive probability by the best-response mix and that yields a lower expected payoff against σ_{-i}. If there is more than one, focus on the one that yields the lowest expected payoff. Suppose Player i drops that (low-yield) pure strategy from the mix, and assigns its probability to one of the other (higher-yield) strategies in the mix. This must raise Player i's expected payoff (just as dropping the player with the lowest batting average on a team must raise the team average). But then the original mixed strategy cannot have been a best response: it does not do as well as the new mixed strategy. This is a contradiction.

An immediate implication of this lesson is that if a mixed strategy forms part of a Nash Equilibrium then each pure strategy in the mix must itself be a best response. Hence all the strategies in the mix must yield the same expected payoff. We will use this fact to find mixed-strategy Nash Equilibria.

Two immediate implications of the above lemma (Lemma 2.1) are

Claim 2.3. *(a) Every strictly dominant strategy of $\Delta\mathcal{G}$ is a pure strategy, i.e. a strategy in $\mathcal{G}$.*
(b) If a pure strategy $s_i \in S_i$ is never a best response, then any mixed strategy σ_i with $\sigma_i(s_i) > 0$ is not in Nash equilibrium strategy.

Proof. (a) Suppose σ_i is a strategy in $\Delta\mathcal{G}$ but not in $\mathcal{G}$ (i.e., σ_i is **not** a pure strategy) and σ_i is strictly dominant in $\Delta\mathcal{G}$. Then by Indifference Lemma (Lemma 2.1), there are two strategies $\hat{s}_i \neq \tilde{s}_i$ belonging to $\mathcal{G}$ such that $\sigma_i(\hat{s}_i) > 0$ and $\sigma_i(\tilde{s}_i) > 0$, and for all s_{-i},

$$EU_i(\hat{s}_i, s_{-i}) = EU_i(\tilde{s}_i, s_{-i}) = EU_i(\sigma_i, s_{-i}).$$

Hence, σ_i is not strictly dominant.
(b) Suppose $s_i \in S_i$ is never a best response but there is a mixed strategy Nash equilibrium

σ with $\sigma_i(s_i) > 0$. By the Indifference Lemma (Lemma 2.1), s_i is also a best response to σ_{-i}, contradicting the fact s_i is never a best response. ■

2.6.4.1 Finding Mixed Strategy Nash Equilibrium

Now let us look at some examples and use the Indifference Lemma (2.1) to find the mixed strategy Nash equilibrium.

Example 2.29. *Imagine a match between the two tennis players – Martina Navratilova and Chris Evert. Navratilova at the net has just volleyed a ball to Evert on the baseline, and Evert is about to attempt a passing shot. She can try to send the ball either down the line (DL; a hard, straight shot) or cross court (CC; a softer, diagonal shot). Navratilova must likewise prepare to cover one side or the other. Payoffs in this tennis-point game are given by the fraction of times a player wins the point in any particular combination of passing shot and covering play. A down-the-line passing shot is stronger than a cross court shot and that Evert is more likely to win the point when Navratilova moves to cover the wrong side of the court, suppose Evert is successful with a down-the-line passing shot* 80% *of the time if Navratilova covers cross court; Evert is successful with the down-the-line shot only* 50% *of the time if Navratilova covers down the line. Similarly, Evert is successful with her cross court passing shot* 90% *of the time if Navratilova covers down the line. This success rate is higher than when Navratilova covers cross court, in which case Evert wins only* 20% *of the time. Clearly, the fraction of times that Navratilova wins this tennis point is just the difference between* 100% *and the fraction of time that Evert wins. We represent the payoffs in the payoff matrix (Figure 2.44)*

		Navratilova	
		DL	CC
Evert	DL	50, 50	80, 20
	CC	90, 10	20, 80

Figure 2.44: Finding Mixed Strategy N.E.: Tennis Game

Observe, each player is aware that she must not give any indication of her planned action to her opponent, knowing that such information will be used against her. Navratilova would move to cover the side to which Evert is planning to hit or Evert would hit to the side that Navratilova is not planning to cover. Both must act in a fraction of a second, and both are equally good at concealing their intentions until the last possible moment; therefore their actions are effectively simultaneous, and we can analyze the point as a two-player simultaneous-move game. As it can be seen there is no pure strategy Nash equilibrium of this game. So, let us try to find out mixed strategy Nash equilibrium of this game.

Suppose Navratilova's mixed strategy assigns q to DL and $1-q$ to CC, that is $\sigma_2 = (q, 1-q)$. Then,
Evert's expected payoff from DL against $(q, 1-q) = q \cdot 50 + (1-q) \cdot 80 = 80 - 30q$
Evert's expected payoff from CC against $(q, 1-q) = q \cdot 90 + (1-q) \cdot 20 = 70q + 20.$

Let Evert's mixed strategy assigns p to DL and $1-p$ to CC. Then, from the above it is clear that her best response is

$$p = \begin{cases} \{1\} & \text{if } q < .6 \\ \{p : 0 \leq p \leq 1\} & \text{if } q = .6 \\ \{0\} & \text{if } q > .6. \end{cases}$$

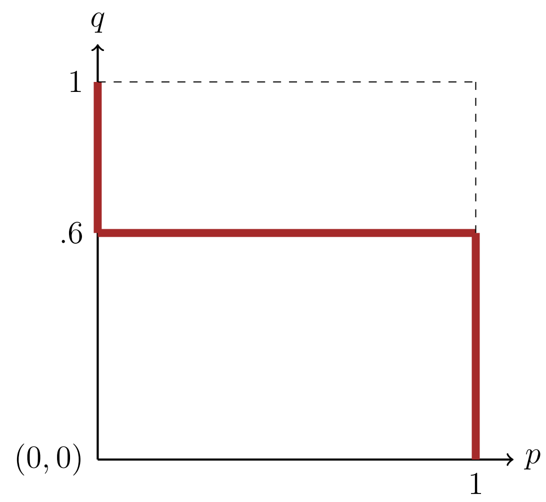


Figure 2.45: Evert's Best Responses in Tennis Game

Observe the application of Indifference Lemma – if Evert mixes DL and CC then both DL and CC must be best responses to $(q, 1-q)$. And, for them to be best responses, they must both yield the same expected payoff. But if these expected payoffs are to be equal, we must have $80 - 30q = 70 + 20q \Rightarrow q = .6$.

To summarize so far, if Evert is mixing on both her strategies in a Nash Equilibrium then both must yield the **same** expected payoff, in which case Navratilova must be mixing with weights $(.6, .4)$.

Notice the trick here: We used the fact that, in equilibrium, *Evert must be indifferent* between the strategies involved in her mix to solve for *Navratilova*'s equilibrium mixed strategy.

Now let's reverse the trick to find Evert's equilibrium mix. Evert's mixed strategy assigns p to DL and $1-p$ to CC, that is $\sigma_1 = (p, 1-p)$. Then

Navratilova's expected payoff from DL against $(p, 1-p) = p \cdot 50 + (1-p) \cdot 10 = 40p + 10$
Navratilova's expected payoff from CC against $(p, 1-p) = p \cdot 20 + (1-p) \cdot 80 = 80 - 60p$.
Hence, Navratilova's best response is

$$q = \begin{cases} \{1\} & \text{if } p > .7 \\ \{q : 0 \leq q \leq 1\} & \text{if } p = .7 \\ \{0\} & \text{if } p < .7. \end{cases}$$

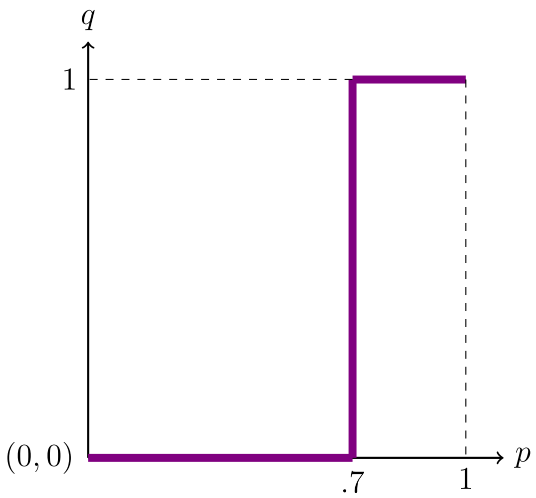


Figure 2.46: Navratilova's Best Responses in Tennis Game

Observe again the application of Indifference Lemma, if there is an equilibrium in which Navratilova mixes on both DL and CC, then both DL and CC must be best responses to whatever Evert is doing. But for them both to be best responses, they must both yield Navratilova the same expected payoff and that gives us $p = .7$ at which Navratilova mixes.

To summarize, if Navratilova is mixing on both her strategies in a Nash Equilibrium then both must yield the same expected payoff, in which case Evert must be mixing with weights $(.7, .3)$. We use the fact that, in equilibrium, Navratilova must be indifferent between the strategies involved in her mix to solve for Evert's equilibrium mixed strategy. We superimpose these two best responses and get Figure 2.47

We argue that this mixed strategy profile $\big((.7, .3), (.6, .4)\big)$ is a Nash equilibrium: We provide two arguments one involving best responses and the other by checking whether any player has any profitable deviation (note that these two arguments are equivalent – why?).

(a) Each player's mix is a best response to the other's choice, so the pair constitutes a Nash equilibrium in mixed strategies.

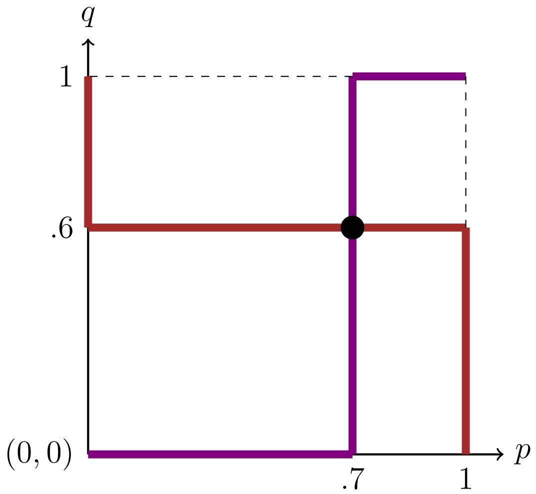


Figure 2.47: Mixed Strategy Nash Equilibrium in Tennis Game

(b) We constructed the equilibrium so that, given Navratilova's mix, $(.6, .4)$, each of Evert's pure strategies, DL and CC yields the same expected payoff. But, in that case, any mix of those pure strategies (including the equilibrium mix itself) will yield the same expected payoff. So all potential deviations yield the same expected payoff: none are *strictly* profitable. The same argument applies to Navratilova.

Observe the power of Indifference Lemma here, again, all we need to check is whether there is any pure strategy deviation or not because if there is none, then any mixed strategies involving those pure strategies will also not be a profitable unilateral deviation. This reduces our work substantially as we need to check only pure strategy deviation.
The question you may ask now is – what is the intuition? Ok, I understand the way we found the mixes and I can also see that mixed strategy profile is a Nash equilibrium, but why are we solving like this? What is the intuition behind this calculation? Is Evert mixing to make Navratilova indifferent? Below I provide an intuitive explanation.

Suppose, Evert does not mix her strategies with $(.6, .4)$ but instead chooses a mixed strategy $(.75, .25)$. Then Navratilova's expected payoff from covering DL is $.75 \cdot 50 + .25 \cdot 10 = 37.5 + 2.5 = 40$ whereas that from covering CC is $.75 \cdot 20 + .25 \cdot 80 = 15 + 20 = 30$. So, if Evert chooses this $(.75, .25)$ mixture, then Navratilova can exploit that by covering DL. When Navratilova chooses DL to best exploit Evert's $(.75, .25)$ mix, her choice works to Evert's disadvantage. Or in other words, Evert's $(.75, .25)$ mix is open to exploitation by Navratilova. Ideally, Evert would like to find a mix that would be exploitation proof – a mix that would leave Navratilova no obvious choice of pure strategy to use against it. Evert's exploitation-proof mixture must have the property that Navratilova gets the same expected payoff against it by covering DL or CC; it must keep Navratilova indifferent between her two pure strategies. We have used this intuition and used *(opponent's) Indifference property* to compute mixed strategy Nash equilibrium.

2.6.4.2 Comparative Statics

Suppose Navratilova goes under rigorous training and her skill to cover down the line (DL) increases and now she becomes successful to cover that 70% of the time. The changed payoff matrix is given in Figure 2.48.

		Navratilova	
		DL	CC
Evert	DL	$30, 70$	$80, 20$
	CC	$90, 10$	$20, 80$

Figure 2.48: Mixed Strategy N.E. in Tennis Game: Comparative Statics

Let again Evert's mix be $(p, 1-p)$ and Navratilova's mix be $(q, 1-q)$. The question we now ask is – what is the effect of this on equilibrium mix $\sigma \equiv \big((p, 1-p), (q, 1-q)\big)$. We will solve mathematically, but let us first try to answer this intuitively. Observe there are two effects:

(a) **Direct Effect:** Navratilova covers down the line more often (this increases q).

(b) **Strategic Effect:** Evert knows this and hence, she hits direct shot less often, so Navratilova covers that less often (q goes down).

So, there is no obvious intuitive answer – whether the new q would be higher than the old one depends on these two opposing effects. If the direct effect is stronger then the new q will be higher otherwise it will be lower. Let us solve mathematically then.

Again to find the new q for Navratilova we use Evert's payoff:
Evert's expected payoff from DL against $(q, 1-q) = q \cdot 30 + (1-q) \cdot 80 = 80 - 50q$
Evert's expected payoff from CC against $(q, 1-q) = q \cdot 90 + (1-q) \cdot 20 = 70q + 20$.
Then, Evert's best response is

$$p = \begin{cases} \{1\} & \text{if } q < .5 \\ \{p : 0 \leq p \leq 1\} & \text{if } q = .5 \\ \{0\} & \text{if } q > .5. \end{cases}$$

Now, let us find out the equilibrium p mix. For that we need to use Navratilova's payoff.
Navratilova's expected payoff from DL against $(p, 1-p) = p \cdot 70 + (1-p) \cdot 10 = 60p + 10$
Navratilova's expected payoff from CC against $(p, 1-p) = p \cdot 20 + (1-p) \cdot 80 = 80 - 60p$.
Hence, Navratilova's best response is

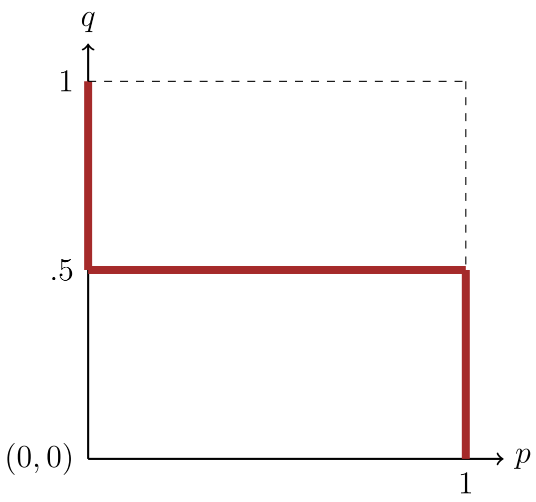


Figure 2.49: Evert's New Best Responses in Tennis Game

$$q = \begin{cases} \{1\} & \text{if } p > \dfrac{7}{12} \\ \{q : 0 \leq q \leq 1\} & \text{if } p = \dfrac{7}{12} \\ \{0\} & \text{if } p < \dfrac{7}{12}. \end{cases}$$

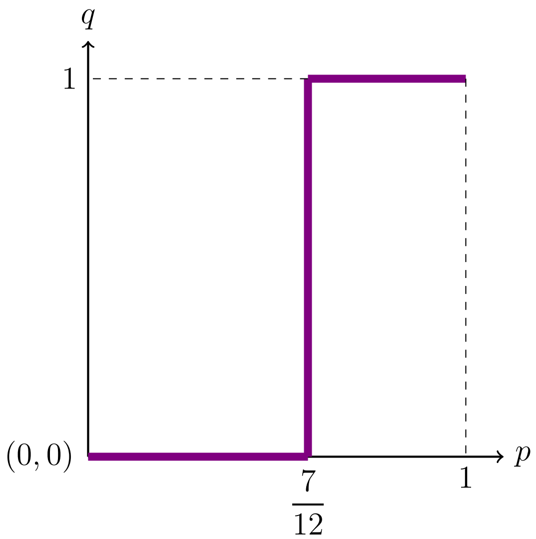


Figure 2.50: Navratilova's New Best Responses in Tennis Game

Hence, the new equilibrium mix is $\left((\frac{7}{12}, \frac{5}{12}), (.5, .5)\right)$ (see figure 2.51).
Both q and p have gone down that is the strategic effect dominates the direct effect.

Note that if all we want to do is to find a mixed strategy Nash equilibrium of a game

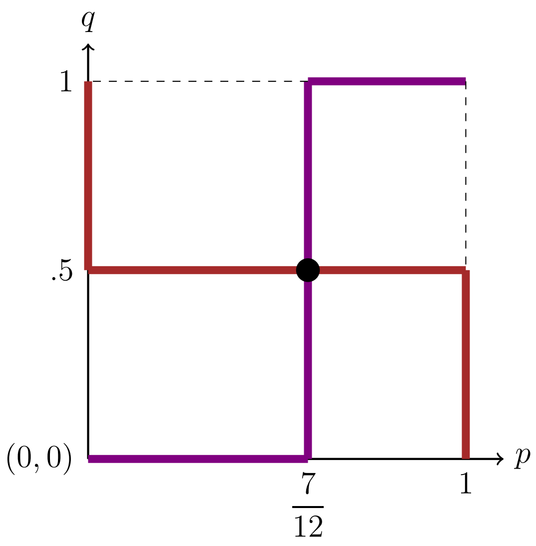


Figure 2.51: New Mixed Strategy Nash Equilibrium in Tennis Game

where each player has just two pure strategies, we don't have to go through the detailed construction of best-response curves, graph them, and look for their intersection. We can write down the exploitation-proofness equations for each player's mixture and solve them. If the solution has both probabilities between 0 and 1, we have found what you want. If the solution includes a probability that is negative, or greater than 1, then the game does not have a mixed-strategy equilibrium; you should go back and look for a pure-strategy equilibrium.

2.6.4.3 Nash Equilibrium as a System of Beliefs and Responses

When the moves in a game are simultaneous, neither player can respond to the other's actual choice. Instead, each takes her best action in light of what she believes the other might be choosing at that instant. We have interpreted Nash equilibrium as a configuration where such beliefs are correct, so each chooses her best response to the actual actions of the other. This concept proved useful for understanding the structures and outcomes of many important types of games.

However, till now we have considered only pure-strategy Nash equilibria. Therefore, a hidden assumption went almost unremarked – namely, that each player was sure or confident in her belief that the other would choose a particular pure strategy. Now that we are considering more general mixed strategies, the concept of belief requires a corresponding reinterpretation.

Players may be unsure about what others might be doing. In the Battle of Sexes game, the woman might be unsure whether the man would choose football match (F) or movies (M), and her belief might be that there was a 50–50 chance that he would go to either one. And in the tennis example, Evert might recognize that Navratilova was trying to keep her guessing and would therefore be unsure of which of the available actions Navratilova would play. Now we develop this idea more fully.

It is important, however, to distinguish between being unsure and having incorrect beliefs. For example, in the tennis example, Navratilova cannot be sure of what Evert is choosing on any one occasion. But she can still have correct beliefs about Evert's mixture – namely, about the probabilities with which Evert chooses between her two pure strategies. Having correct beliefs about mixed actions means knowing or calculating or guessing the correct probabilities with which the other player chooses from among her underlying basic or pure actions. In the equilibrium of our original example (Example 2.29), it turned out that Evert's equilibrium mixture is $(.7, .3)$. If Navratilova believes that Evert will play DL with .7 probability and CC with .3 probability, then her belief, although uncertain, will be *correct in equilibrium.*

Thus, we have an alternative and mathematically equivalent way to define Nash equilibrium in terms of beliefs: Each player forms beliefs about the probabilities of the mixture that the other is choosing and chooses her own best response to this. A Nash equilibrium in mixed strategies occurs when the beliefs are correct, in the sense just explained.

Next, we consider mixed strategies and their Nash equilibria where players payoffs are not competitor.[10] In such games, there is no general reason that the other player's pursuit of her own interests should work against your interests. Therefore, it is not in general the case that a player would want to conceal his intentions from the other player, and there is no general argument in favor of keeping the other player guessing. However, because moves are simultaneous, each player may still be subjectively unsure of what action the other is taking and therefore may have uncertain beliefs that in turn lead her to be unsure about how he should act. This can lead to mixed strategy equilibria, and their interpretation in terms of subjectively uncertain but correct beliefs proves particularly important.

Next we consider *Battle of Sexes* game. Here since players' interests coincide to some extent, the fact that the man will "exploit" the woman's systematic choice, if any, does not work to her disadvantage. In fact, the players are able to coordinate if each can rely on the other's choosing strategies systematically, random choices only increase the risk of coordination failure. As a result, mixed-strategy equilibria have a weaker rationale, and sometimes no rationale at all in these type of games.

Finding Mixed Strategy in the Battle of Sexes Game Recall the payoff matrix (drawn again in Figure 2.52) of the game. Let the woman's mix be $(p, 1-p)$ and the man's mix be $(q, 1-q)$. In order to find out the equilibrium q we use the woman's payoff:
The woman's expected payoff from F against $(q, 1-q) = q \cdot 2 + (1-q) \cdot 0 = 2q$
The woman's expected payoff from M against $(q, 1-q) = q \cdot 0 + (1-q) \cdot 1 = 1-q$

[10] We call such games strictly competitive games which we define in Section 2.8

		Man	
		F	M
Woman	F	$2, 1$	$0, 0$
	M	$0, 0$	$1, 2$

Figure 2.52: Battle of Sexes

Hence, the woman's best response is

$$p = \begin{cases} \{1\} & \text{if } q > \dfrac{1}{3} \\ \{p : 0 \leq p \leq 1\} & \text{if } q = \dfrac{1}{3} \\ \{0\} & \text{if } q < \dfrac{1}{3}. \end{cases}$$

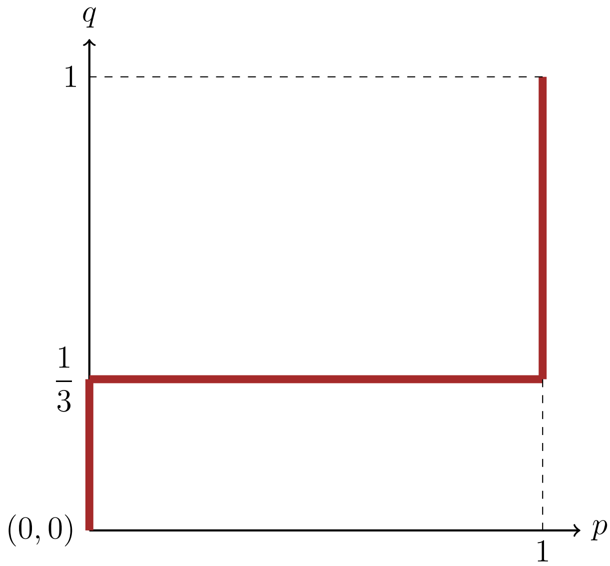


Figure 2.53: The Woman's Best-Responses in the Battle of the Sexes Game

Similarly, in order to find out the equilibrium p we use the man's payoff:
The man's expected payoff from F against $(p, 1-p) = p \cdot 1 + (1-p) \cdot 0 = p$
The man's expected payoff from M against $(p, 1-p) = p \cdot 0 + (1-p) \cdot 2 = 2(1-p)$

Hence, the man's best response is

$$q = \begin{cases} \{1\} & \text{if } p > \dfrac{2}{3} \\ \\ \{q : 0 \leq q \leq 1\} & \text{if } p = \dfrac{2}{3} \\ \\ \{0\} & \text{if } p < \dfrac{1}{3}. \end{cases}$$

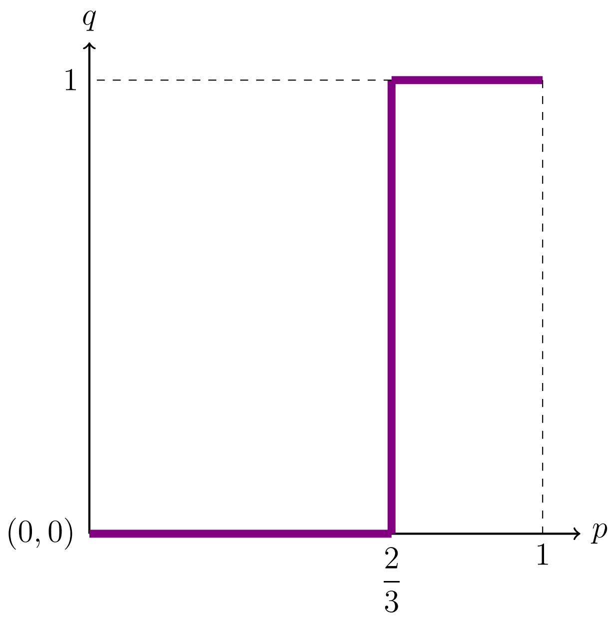


Figure 2.54: The Man's Best-Responses in the Battle of the Sexes Game

Hence, there are three Nash equilibria in this game: $\big((1,0),(1,0)\big)$, $\big((0,1),(0,1)\big)$, and $\big((\frac{2}{3},\frac{1}{3}),(\frac{1}{3},\frac{2}{3})\big)$ where first two are pure strategies Nash equilibria (see Figure 2.55).

Observe, the way we have solved for the third equilibrium. Here, both the players are indifferent between their pure strategies – it is *as if* each player is choosing his/her strategies to keep the other player indifferent. We emphasize the "as if" because in this game, the woman has no reason to keep the man indifferent and vice versa. The outcome is merely a property of the equilibrium.

Still, the general idea is worth remembering: in a mixed-strategy Nash equilibrium, each person's mixture probabilities keep the other player indifferent between his pure strategies.

Now let us compare the expected payoffs. In the pure strategy equilibria payoffs are $(2,1)$ at $\big((1,0),(1,0)\big)$ and $(1,2)$ at $\big((0,1),(0,1)\big)$. In the mixed strategy equilibrium, the mixed strategy payoff of the woman is $\frac{2}{3}\cdot\frac{1}{3}\cdot 2+\frac{2}{3}\cdot\frac{2}{3}\cdot 0+\frac{1}{3}\cdot\frac{1}{3}\cdot 0+\frac{1}{3}\cdot\frac{2}{3}\cdot 1=\frac{2}{3}$.

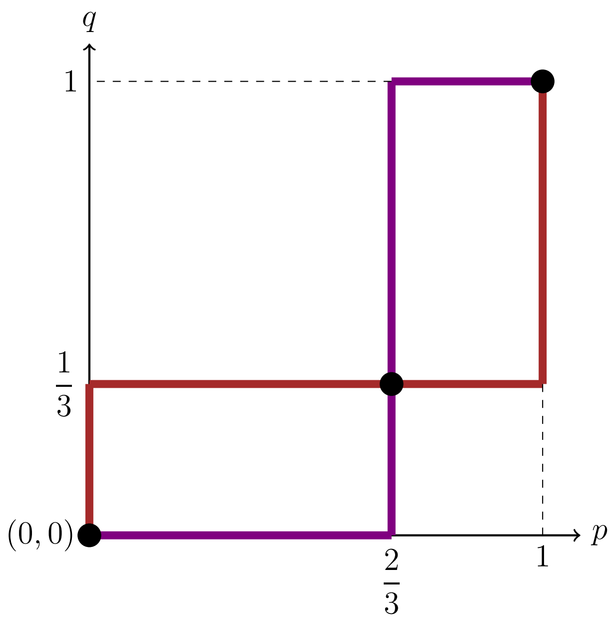


Figure 2.55: Computing all the Nash equilibria of the Battle of the Sexes Game

Similarly, the man's expected utility is $\frac{2}{3} \cdot \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot \frac{2}{3} \cdot 2 = \frac{2}{3}$. Thus each of them gets lower than what he/she was getting at the worse pure strategy equilibrium. But why?

The reason the two players fare so badly in the mixed-strategy equilibrium is that when they choose their actions independently and randomly, they create a significant probability of going to different places; when that happens, they do not meet, and each gets a payoff of 0. The probability of this happening when both are using their equilibrium mixtures is $\frac{2}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{5}{9} > \frac{1}{2}$, that is they fail to meet more than half of the time.

Now we provide another interpretation of mixed strategies: Equilibrium mix can be interpreted as the proportion of population who are choosing one action. Let us consider the following example.

Example 2.30. *Consider a representative potential tax payer from a large population. He is deciding whether to pay tax honestly (H) or cheat (C). At the same time an auditor is deciding whether to audit (A) or not (N). Utility from paying tax is normalized to zero. On the other hand, if the person does not pay tax and get caught he has to pay a fine, let utility from that be* -10 *whereas if he does not get caught let his utility be* 4. *The auditor is the happiest when he does not audit and the person pays tax or he audits and finds the person was actually cheating. Let the auditor's utility in these two cases be* 4. *Let the auditor's utility when he audits but finds that the person was paying tax be* 2. *Finally, let the auditor's utility be* 0 *when he does not audit and the taxpayer also does not pay tax. The payoff matrix is given in Figure 2.56*

Observe, there is no pure strategy Nash equilibrium. Now let us find out the mixed

		Taxpayer	
		H	C
Auditor	A	$2, \underline{0}$	$\underline{4}, -10$
	N	$\underline{4}, 0$	$0, \underline{4}$

Figure 2.56: Auditing Tax

strategy Nash equilibrium. Let the mix be $\big((p, 1-p), (q, 1-q)\big)$.

In order to find out auditor's mix we use the taxpayer's indifference condition: $0 = p \cdot (-10) + (1-p) \cdot 4 \Rightarrow p = \frac{2}{7}$. And, similarly we find equilibrium q from the auditor's indifference condition: $2 \cdot q + 4 \cdot (1-q) = 4 \cdot q + 0 \cdot (1-q) \Rightarrow q = \frac{2}{3}$. Hence, the Nash equilibrium is $\big((\frac{2}{7}, \frac{5}{7}), (\frac{2}{3}, \frac{1}{3})\big)$.

We interpret this result as follows. At equilibrium two-third of the potential tax payers actually pay taxes and the auditor remains indifferent between auditing and not auditing. On the other hand, the auditor audits with probability $\frac{2}{7}$.

Now suppose the government wants to increase tax compliance and to do that the government increases the fine to 20. The new payoff matrix is given in Figure 2.57

		Taxpayer	
		H	C
Auditor	A	$2, \underline{0}$	$\underline{4}, -20$
	N	$\underline{4}, 0$	$0, \underline{4}$

Figure 2.57: Auditing Tax: Comparative Statics

Now let us find out the effect of this change. Again we find auditor's mix from the taxpayer's indifference condition: $0 = p \cdot (-20) + (1-p) \cdot 4 \Rightarrow p = \frac{1}{6} < \frac{2}{7}$. However, observe as auditor's payoff has not changed equilibrium q does not change. That is this change in policy has no effect on tax compliance, this only increases the probability of auditing.

What is the intuition here? To increase the tax compliance rate, what should the government actually do?

2.6.5 Rationalizability and Nash Equilibrium

Every strategy that is a part of a Nash equilibrium is rationalizable because each player's strategy in Nash equilibrium can be justified by the Nash equilibrium strategies of the other players. Thus, as a general matter, the Nash equilibrium concept offers at least as sharp a prediction as does the rationalizability concept. In fact, it often offers a *much* sharper prediction. To see this, note that in Example 2.28 a_1, a_3 b_1 and b_3 are rationalizable but are not part of any Nash equilibrium because at these strategies players' beliefs about each other's play are not correct.

In some games, iterated elimination of never-best-response strategies can narrow things down all the way to Nash equilibrium. So, rationalizability can take us all the way to Nash equilibrium. Note we said *can*, not must. But if it does, that is useful because in these games we can strengthen the case for Nash equilibrium by arguing that it follows purely from the players' rational thinking about each other's thinking. Observe that one such game is Cournot competition.

Exercise 2.16. *Find out the rationalizable strategies and Nash equilibria of the following games:*

Player 1 \ Player 2	L	C	R
T	$0,7$	$2,5$	$7,0$
M	$5,2$	$3,3$	$5,2$
B	$7,0$	$2,5$	$0,7$

Player 1 \ Player 2	L	C	R	S
T	$0,7$	$2,5$	$7,0$	$0,1$
M	$5,2$	$3,3$	$5,2$	$0,1$
B	$7,0$	$2,5$	$0,7$	$0,1$
W	$0,0$	$0,-2$	$0,0$	$10,-1$

Hint. In the first game, all the strategies are rationalizable. Argue that properly. In the second game, check if S is a best response. Then is W a best response? If yes, is that a best response to never a best response?

Exercise 2.17. *Show that there are no mixed-strategy Nash equilibria (that is they have only pure strategy Nash equilibria) in Prisoner's Dilemma game, in Example 2.5 and in Example 2.6. Comment.*

Exercise 2.18. *Solve for the mixed-strategy Nash equilibria of all the classic games mentioned in Exercise 2.2.*

Exercise 2.19. *Solve for the mixed-strategy Nash equilibria of the following games:*

Player 1 \ Player 2	L	C	R
T	$2,0$	$1,1$	$4,2$
M	$3,4$	$1,2$	$2,3$
B	$1,3$	$0,2$	$3,0$

Player 1 \ Player 2	L	R
T	$2,1$	$0,2$
B	$1,2$	$3,0$

Exercise 2.20. (Penalty Shot) *Player 1 has to take a soccer penalty shot to decide the game. She can shoot Left (L), Middle (M), or Right (R). Player 2 is the goalie. He can dive to the left (l), middle (m), or right (r). Actions are chosen simultaneously. The payoffs (which here are the probabilities in tenths of winning) are as follows.*

Player 2

		l	m	r
	L	$4,6$	$7,3$	$9,1$
Player 1	M	$6,4$	$3,7$	$6,4$
	R	$9,1$	$7,3$	$4,6$

(a) *For each player, is any strategy dominated by another (pure) strategy?*

(b) *For what beliefs about player 1's strategy is m a best response for player 2? For what beliefs about player 2's strategy is M a best response for player 1? [Hint: you do not need to draw a 3-dimensional picture! In fact, check that $(.5, 0, .5)$ mixed strategy of Player 1 strictly dominates M].*

(c) *Suppose player 2 "puts himself in player 1's shoes" and assumes that player 1, what ever is her belief, will always choose a best-response to that belief. Should player 2 ever choose m?*

(d) *Show that this game does not have a (pure-strategy) Nash Equilibrium.*

Exercise 2.21. (Partnerships Revisited.) *Recall the partnership game we discussed in class. Two partners jointly own a firm and share equally in its revenues. Each partner individually decides how much effort to put into the firm. The firm's revenue is given by $4(s_1 + s_2 + bs_1s_2)$ where s_1 and s_2 are the efforts of the lawyers 1 and 2 respectively. The parameter $b > 0$ reflects the synergies between their efforts: the more one works, the more productive is the other. Assume that $0 \leq b \leq \frac{1}{4}$, and that each effort level $S_i = [0, 4]$. The payoffs for partners 1 and 2 are:*

$$u_1(s_1, s_2) = \frac{1}{2}[4(s_1 + s_2 + bs_1s_2)] - s_1^2$$

$$u_2(s_1, s_2) = \frac{1}{2}[4(s_1 + s_2 + bs_1s_2)] - s_2^2$$

respectively, where the s_i^2 terms reflect the cost of effort. (Notice that the cost of providing another unit of effort is increasing in the amount of effort already provided). Assume the firm has no other costs.

(a) *Find out the rationalizable strategies.*

(b) *Suppose that the partners both agree to work the same amount as each other, and that they write a contract specifying that amount. What common amount of effort s^{**} should they agree each to supply to the firm if their aim is to maximize revenue net of total effort costs. How does this amount compare to the rationalizable effort levels you found in (a). Give a brief intuition for this comparison. [Hint: for the intuition, it may help to consider the special case $b = 0$.]*

(c) *Suppose now that the contract is only binding on partner 2. That is, partner 2 has to provide the effort level* s^{**} *you found from part* (b)*, but partner 1 is free to choose any effort level between* 0 *and* 4*. What effort level will partner 1 choose? How does this amount compare to* s_1^* *and* s_1^{**}*. Give a brief intuition for you answer.*

(d) *Return to the basic game we discussed in class, but now assume that* $b = -\frac{1}{4}$*; that is, the partners' efforts have negative synergies. Solve for the best-response functions in this case, and draw the best-response diagram. Find the set of rationalizable strategies. Again, compare these effort levels with those that the partners would choose if they could contract to provide the same amount as each other.*

Relevant Parts of the Reference Book: Chapter 11.

2.7 The Maxmin Concept

Consider a game shown in figure 2.58. There is a unique Nash equilibrium of this game: (B, R) – verify this. But, will Player 1 play strategy B? One can imagine Player 1 hesitating to choose B: what if Player 2 were to choose L (whether by accident, due to irrationality, or for any other reason)? Given that the result (B, L) is catastrophic for Player 1, he may prefer strategy T, guaranteeing a payoff of only 2 (compared to the equilibrium payoff of 3), but also guaranteeing that he will avoid getting -100 instead. For Player 2 also, strategy R may be bad, what if Player 1 decides to play T? On the other hand, strategy L can guarantee him a payoff of 0.

		Player 2	
		L	R
	T	$2, 1$	$2, -20$
Player 1	M	$3, 0$	$-10, 1$
	B	$-100, 2$	$3, 3$

Figure 2.58: Motivation for Maxmin

The main message of the example is that sometimes players may choose to play strategy to guarantee themselves some safe level of payoff without assuming anything about the rationality level of other players. In particular, we consider the case where every player believes that the other players are adversaries and are here to punish him - this is a very pessimistic view of the opponents. In such a case, what can a player guarantee for himself?

If Player i chooses a strategy $s_i \in S_i$ in a game, then the worst payoff he can get is

$$\min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}).$$

Player i can choose the strategy s_i that maximizes this value.

Definition 2.17. *The* **maxmin** *value for Player i in a strategic form game $\mathcal{G} \equiv \langle N, \{S_i\}_{i=1}^N, \{u_i\}_{i=1}^N \rangle$ is given by*

$$\underline{v_i} := \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}).$$

Any strategy that guarantees Player i a value of $\underline{v_i}$ is called a **maxmin strategy**.

Observe if $\hat{s}_i$ is a maxmin strategy for Player i, then it satisfies

$$\min_{s_{-i} \in S_{-i}} u_i(\hat{s}_i, s_{-i}) \geq \min_{s_{-i} \in S_{-i}} u_i(s_i^{'}, s_{-i}) \qquad \forall s_i^{'} \in S_i.$$

This also means that $u_i(\hat{s}_i, s_{-i}) \geq \underline{v_i}$ for all $s_{-i} \in S_{-i}$.

Let us now find out the maxmin value and maxmin strategy for each player, and correspondingly the outcome of the game given in Figure 2.58: $\underline{v_1} = 2$ and $\underline{v_2} = 0$. Strategy T is a maxmin strategy for Player 1 and strategy L is a maximin strategy for Player 2. Hence, when players play their maxmin strategy, the outcome of the game is $(2, 1)$.

		Player 2: L	Player 2: R	$\min_{s_2 \in S_2} u_1(s_1, s_2)$
Player 1	T	$2, 1$	$2, -20$	2
	M	$3, 0$	$-10, 1$	-10
	B	$-100, 2$	$3, 3$	-100
	$\min_{s_1 \in S_1} u_2(s_1, s_2)$	0	-20	

Figure 2.59: The Game in Figure 2.58 with Maxmin Value of Each Player

However, there can be more than one maxmin strategies in a game, in which case no unique outcome can be predicted. Consider the example in Figure 2.60. The maxmin strategy for Player 1 is B. But Player 2 has two maxmin strategies $\{L, R\}$, both giving a payoff of 1. Depending on which maxmin strategy Player 2 plays the outcome can be $(2, 3)$ or $(1, 1)$.

It is clear that if a player has a weakly dominant strategy, then it is a maxmin strategy – it guarantees him the best possible payoff irrespective of what other agents are playing. Hence, if every player has a weakly dominant strategy, then the vector of weakly dominant strategies constitute a vector of maxmin strategies. This was true, for

		Player 2		
		L	R	$\min_{s_2 \in S_2} u_1(s_1, s_2)$
Player 1	T	$3, 1$	$0, 4$	0
	B	$2, 3$	$1, 1$	1
	$\min_{s_1 \in S_1} u_2(s_1, s_2)$	1	1	

Figure 2.60: A Game with More than One Maxmin Strategy

instance, in the example involving the second-price sealed-bid auction. Further, if there are strictly dominant strategies for each player (note such strategy must be unique for each player), then the vector of strictly dominant strategies constitute a unique vector of maxmin strategies.

The following lemma shows that a Nash equilibrium of a game guarantees the maxmin value for every player.

Lemma 2.2. *Every Nash equilibrium s^* of a strategic form game satisfies*

$$u_i(s^*) \geq \underline{v}_i \quad \forall i \in N.$$

Proof. For any Player i and for every $s_i \in S_i$, we know that

$$u_i(s_i, s^*_{-i}) \geq \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}).$$

By definition, $u_i(s^*_i, s^*_{-i}) = \max_{s_i \in S_i} u_i(s_i, s^*_{-i})$. Combining with the above inequality, we get $u_i(s^*_i, s^*_{-i}) = \max_{s_i \in S_i} u_i(s_i, s^*_{-i}) \geq \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) = \underline{v}_i$. ■

2.8 Zero-Sum Game

As we have seen, the Nash equilibrium and the maxmin are two different concepts that reflect different behavioural aspects: the first is an expression of stability, while the second captures the notion of security. Despite the different roots of the two concepts, there are cases in which both lead to the same results. A special case where this occurs is in the class of **two-player zero-sum games**, which is the subject of this section. Let us consider the following example.

Example 2.31. *Player 1 has three strategies: $S_1 = \{T, M, B\}$ and Player 2 has three strategies $S_2 = \{L, C, R\}$. The payoffs are given in Figure 2.61.*

In the game in Example 2.31, for each pair of strategies the sum of the payoffs that the two players receive is zero. In other words, in any possible outcome of the game the payoff one player receives is exactly equal to the payoff the other player has to pay.

Player 2

Player 1		L	C	R
	T	$3, -3$	$-5, 5$	$-2, 2$
	M	$1, -1$	$4, -4$	$1, -1$
	B	$6, -6$	$-3, 3$	$-5, 5$

Figure 2.61: A Two-Player Zero-Sum Game

Definition 2.18. *A two-player game is a zero-sum game if for each pair of strategies* (s_1, s_2) *one has*

$$u_1(s_1, s_2) + u_2(s_1, s_2) = 0.$$

In other words, a two-player game is a zero-sum game if each player gains what the other player loses. It is clear that in such a game the two players have diametrically opposed interests.

More generally, the idea is that the players' interests are in complete conflict. Such conflict arises when players are dividing up any fixed amount of possible gain. Because the available gain need not always be exactly 0, the term **constant-sum game** is often substituted for zero-sum game. Another commonly used terminology is **strictly competitive game.** A two-player, strictly competitive game is a two-player game with the property that for every two strategy profiles $s, s^{'} \in S$, $u_1(s) > u_1(s^{'})$ if and only if $u_2(s^{'}) > u_2(s)$.

Let us now turn to the study of two-player zero-sum games. Since the payoffs u_1 and u_2 satisfy $u_1 + u_2 = 0$, we can confine our attention to one function, $u_1 = u$, with $u_2 = -u$. The function u will be termed the payoff function of the game, and it represents the payment that Player 2 makes to Player 1. Note that this creates an artificial asymmetry (albeit only with respect to the symbols being used) between the two players: Player 1 seeks to maximize $u(s)$ (his payoff) and Player 2 is trying to minimize $u(s)$, which is what he is paying (since his payoff is $-u(s)$).
The game in Example 2.31 can therefore be represented as shown in Figure 2.62.

Player 2

Player 1		L	C	R
	T	3	-5	-2
	M	1	4	1
	B	6	-3	-5

Figure 2.62: The payoff function u of the zero-sum game in Example 2.31

The game of Matching Pennies (Example 2.10) can also be represented as a zero-sum game (see Figure 2.63).

Consider now the maxmin values of the players in a two-player zero-sum game. Player

Player 2

Player 1		Heads	Tails
	Heads	-1	1
	Tails	1	-1

Figure 2.63: The payoff function u of the game Matching Pennies

1's maxmin value is given by

$$\underline{v}_1 = \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2)$$

and Player 2's maxmin value is

$$\underline{v}_2 = \max_{s_2 \in S_2} \min_{s_1 \in S_1} (-u(s_1, s_2)) = -\min_{s_2 \in S_2} \max_{s_1 \in S_1} (u(s_1, s_2)).$$

Denote

$$\underline{v} := \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2)$$

$$\bar{v} := \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2)$$

The value $\underline{v}$ is called the maxmin value of the game, and $\bar{v}$ is called the minmax value. Player 1 can guarantee that he will get at least $\underline{v}$, and Player 2 can guarantee that he will pay no more than $\bar{v}$. A strategy of Player 1 that guarantees $\underline{v}$ is termed a maxmin strategy. A strategy of Player 2 that guarantees $\bar{v}$ is called a minmax strategy.

We next calculate the maxmin value and minmax value in various examples of games. In Example 2.31, $\underline{v} = 1$ and $\bar{v} = 1$. In other words, Player 1 can guarantee that he will get a payoff of at least 1 (using the maxmin strategy M), while Player 2 can guarantee that he will pay at most 1 (by way of the minmax strategy R).

Player 2

Player 1		L	C	R	$\min_{s_2 \in S_2} u_1(s_1, s_2)$
	T	3	-5	-2	-5
	M	1	4	1	1
	B	6	-3	-5	-5
	$\max_{s_1 \in S_1} u_2(s_1, s_2)$	6	4	1	

Figure 2.64: Game in Example 2.31 with the maxmin and minmax values

Now consider the game shown in Figure 2.65. In this game $\underline{v} = 0$ but $\bar{v} = 3$. Player 1 cannot guarantee that he will get a payoff higher than 0 (which he can guarantee using his maxmin strategy B) and Player 2 cannot guarantee that he will pay less than 3 (which he can guarantee using his minmax strategy L).

		Player 2		
		L	R	$\min_{s_2 \in S_2} u_1(s_1, s_2)$
Player 1	T	-2	5	-2
	B	3	0	0
	$\min_{s_1 \in S_1} u_2(s_1, s_2)$	3	5	

Figure 2.65: A Game with Maxmin and Minmax Values

Finally, look again at the game of Matching Pennies (Figure 2.66). In this game, $\underline{v} = -1$ and $\bar{v} = 1$. Neither of the two players can guarantee a result that is better than the loss of one dollar (the strategies H and T of Player 1 are both maxmin strategies, and the strategies H and T of Player 2 are both minmax strategies).

		Player 2		
		H	T	$\min_{s_2 \in S_2} u_1(s_1, s_2)$
Player 1	H	-1	1	-1
	T	1	-1	-1
	$\min_{s_1 \in S_1} u_2(s_1, s_2)$	1	1	

Figure 2.66: Matching Pennies with the maxmin and minmax values

As these examples indicate, the maxmin value $\underline{v}$ and the minmax value $\bar{v}$ may be unequal, but it is always the case that $\underline{v} \leq \bar{v}$. The inequality is clear from the definitions of the maxmin and minmax: Player 1 can guarantee that he will get at least $\underline{v}$, while Player 2 can guarantee that he will not pay more than $\bar{v}$. As the game is a zero-sum game, the inequality $\underline{v} \leq \bar{v}$ must hold.[11]

Definition 2.19. *A two-player game has a* **value** *if $\underline{v} = \bar{v}$. The quantity $v := \underline{v} = \bar{v}$ is then called the value of the game. Any maxmin and minmax strategies of Player 1 and Player 2 respectively are then called* **optimal strategies.**

We end this chapter by stating the relationship between optimal strategies and Nash equilibrium (the proof is beyond the scope of this course).

Lemma 2.3. *If a two-player zero-sum game has a value v, and if s_1^* and s_2^* are optimal strategies of the two players, then $s^* = (s_1^*, s_2^*)$ is a Nash equilibrium with payoff $(v, -v)$. If $s^* = (s_1^*, s_2^*)$ is a Nash equilibrium of a two-player zero-sum game, then the game has a value $v = u(s_1^*, s_2^*)$, and the strategies s_1^* and s_2^* are optimal strategies.*

Exercise 2.22. *Find the maxmin strategy of each candidate in the Voting game (Example 2.4).*

[11] We do not prove this formally.

Exercise 2.23. *Show whether or not the value exists in each of the following games. If the value exists, find it and find all the optimal strategies for each player.*

Player 2

Player 1		L	C	R
	T	1	2	3
	B	4	3	0

Player 2

Player 1		L	R
	T	2	2
	B	1	3

Player 2

Player 1		a	b	d	d
	A	3.5	3	4	12
	B	7	5	6	13
	C	4	2	3	0

Player 2

Player 1		a	b
	A	3	0
	B	2	2
	C	0	3

<u>Relevant Parts of the Reference Book:</u> Chapter 12.

Chapter 3

Dynamic Games of Complete Information

In the examples we examined in chapter 1, such as the the prisoner's dilemma, the battle of sexes, the players choose their actions *simultaneously*. In many games players move sequentially. Game theorists use the concept of *games in extensive form* to model such dynamic situations. The extensive form makes explicit the order in which players move, and what each player knows when making each of his decisions. In this setting, strategies correspond to contingent plans instead of uncontingent actions. In this chapter we restrict attention to games with complete information (i.e., games in which players' payoff functions are common knowledge). We consider some examples first.

Example 3.1. Entry Game *An incumbent faces the possibility of entry by a challenger. The challenger may enter or not. If it enters, the incumbent may either accommodate or fight. Suppose that the best outcome for the challenger is that it enters and the incumbent accommodates, and the worst outcome is that it enters and the incumbent fights, whereas the best outcome for the incumbent is that the challenger stays out, and the worst outcome is that it enters and there is a fight.*

An important feature of this game is that challenger has completely observed what incumbent has done. His action is contingent on what he has observed so far in the game. Such games are called extensive form games with perfect information, i.e., where every player has perfectly observed what has happened so far in the game at every point. The outcomes of the game are realized after the game ends. Players assign payoffs to this terminal stages of the game – this will involve assigning payoffs to every possible sequence of moves in the game. Figure 3.1 depicts the extensive form game using a *game tree*.

We now look at another example where perfect information is absent.

Example 3.2. *Suppose two friends are trying to meet. Friend 1 observes the weather in his city, which is either rain or sunny. Then, he decides to either go to Friend 2's place*

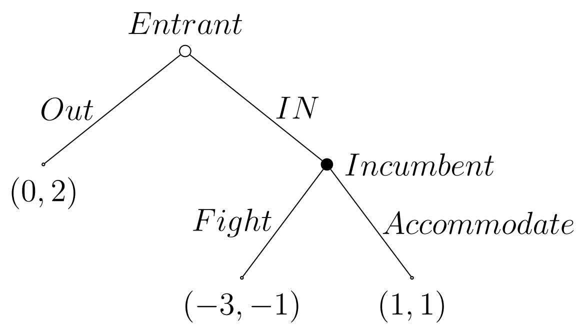


Figure 3.1: The Entry Game

or stay at home. If Friend 1 stays at home, Friend 2 does not do anything and the game ends. If Friend 1 comes to Friend 2's place, she either takes him for dinner or cooks at home.

Crucial here is the fact that Friend 2 *does not observe* the weather in Friend 1's city, which Friend 1 has observed. However, Friend 2 observes whether he has come to her place or not. But Friend 2 does not know if Friend 1 has come from a sunny city or rainy city. In that sense, though the game has sequential nature, the information is *not perfect* in this game.

There is a way to represent this game as an extensive form game with imperfect information. This is done by introducing the dummy player (Nature) who creates the imperfect information. Nature makes the first move by taking either the action "Rainy" or "Sunny". The action of Nature is observed by Friend 1 but not by Friend 2. After observing the action of Nature, Friend 1 takes either of the actions "Stay home" or "Go to Friend 2". Friend 1 can now come to Friend 2 from a Sunny city or a Rainy city. This idea is captured by an *information set*, where a bunch of nodes in the game are combined together to capture Friend 2's uncertainty about where she is in the game. Irrespective of where she is in the game, she observes that Friend 1 has come to her place, and then she chooses one of the actions "go out" or "stay in".

Figure 3.2 shows the extensive form game with information set. The information set of Player 2 is shown in dashed line – it consists of two nodes in the game tree. At this information set, Player 2 does not know if Player 1 has come from a sunny city or rainy city. Each of the possible paths in the game are assigned a payoff for each player. Further, games of imperfect information also specify probabilities/priors of uncertain moves of Nature. These are used to compute expected payoffs on information sets.

3.1 The Extensive Form

The extensive form of a game contains the following information:

1. the set of players

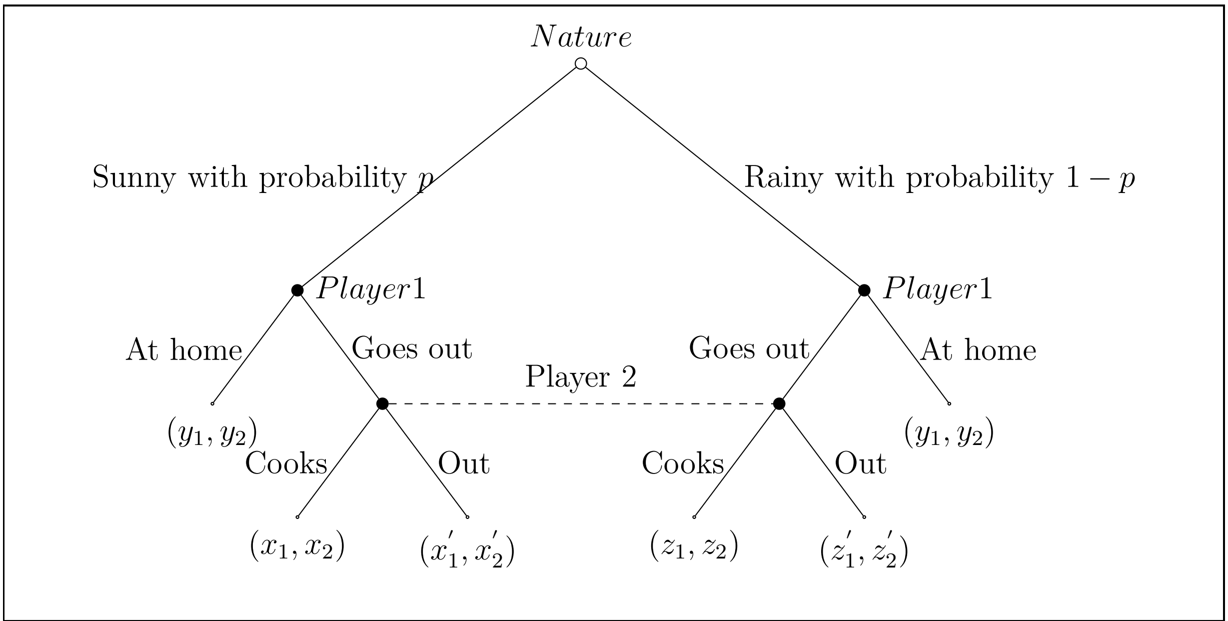


Figure 3.2: Extensive form Game with Imperfect Information

2. the order of moves – i.e. who moves when
3. the players' payoffs as a function of the move that were made
4. what the players' choices are when they move
5. what each player knows when he makes his choices
6. the probability distributions over any exogenous events.

The set of players is denoted by $i \in N$; the probability distributions over exogenous even (point 6) are represented as moves by "Nature". The order of play (point 2) is represented by a *game tree*, T.

3.1.1 Game Tree

Much like an ordinary tree, a game tree starts from a root which is called *initial node* (represented by an open circle). At this initial node one of the players has to make a choice. The various choices available to this player are represented as branches emanating from the initial node. At the end of each branch is another decision node (represented by a solid node), at which another player makes a choice. A game ends at terminal nodes (from which no branches emanate). At the terminal nodes the payoff vectors are displayed.

Consider the game tree in Figure 3.3. a is the initial node. Nodes b, c, d, e are successors of node a. Node b is an immediate successor of node a and the immediate predecessor of node c. Obviously, a given node x is a successor of node y if and only if y is a predecessor of x. Also note that for nodes x, y, and z, if x is a predecessor

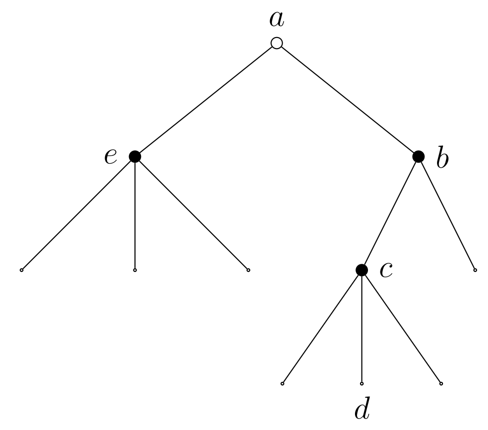


Figure 3.3: Game Tree

of y and y is a predecessor of z, then it must be that x is a predecessor of z. That is, trees have a *transitive precedence relation.*

Observe, every node is a successor of the initial node, and the initial node is the only one with this property.

A *path* through the tree is a sequence of nodes that (a) starts with the initial node, (b) ends with a terminal node, and (c) has the property that successive nodes in the sequence are immediate successors of each other. Each path is one way of tracing through the tree by following branches. For example, in Figure 3.3, one path starts at the initial node, goes to node a and then node b, and ends at node c. We require that terminal nodes exist and that each terminal node completely and uniquely describes a path through the tree. Thus, no two paths should cross. So we assume that each node except the initial node has exactly one immediate predecessor. The initial node has no predecessors.

Remember that branches represent actions that the players can take at decision nodes in the game. We label each branch in a tree with the name of the action that it represents and we assume that multiple branches extending from the same node have different action labels.

Now point 5, the information players have when choosing their actions, is the most subtle of the six points. The information is represented using *information sets.*

Definition 3.1. *An* **information set** *for a player is a collection of decision nodes satisfying:*

(i) the player has the move at every node in the information set, and

(ii) when the play of the game reaches a node in the information set, the player with the move does not know which node in the information set has (or has not) been reached.

Part (*ii*) of this definition implies that the player must have the same set of actions at each decision node in an information set, else the player would be able to infer from the set of actions available that some node(s) had or had not been reached. Figure 3.4(i) is such an example. At decision node a Player 2 has two available actions whereas at decision node b he has three available actions. So, just by observing the number of available actions he is able to infer which decision node he is at. Figure 3.4(ii) also does not make sense as it implies that at some point in the game, the players do not know who is to make a decision.

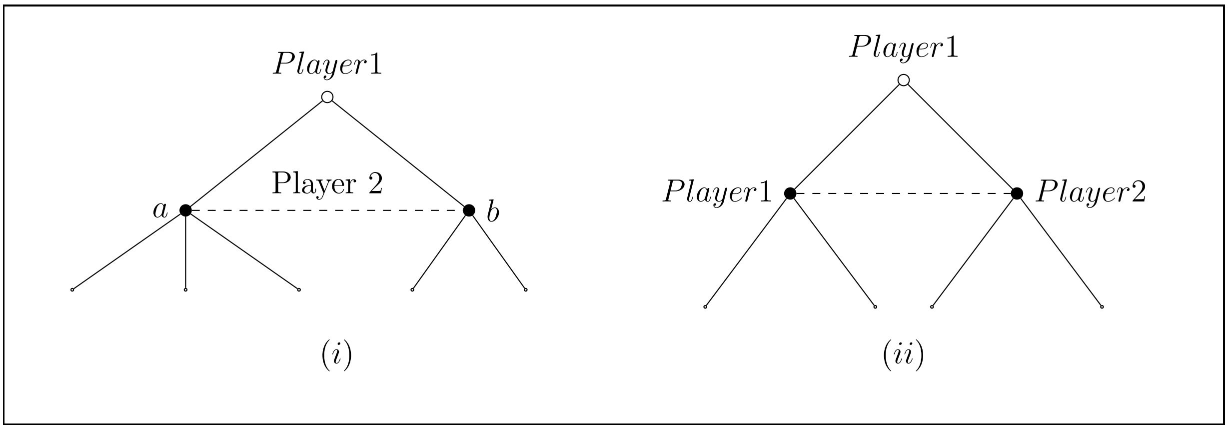


Figure 3.4: Crazy Information Sets

Now that we have defined the notion of an information set, we can offer an alternative definition of the distinction between perfect and imperfect information. We previously defined perfect information to mean that at each move in the game the player with the move knows the full history of the play of the game thus far. An equivalent definition of perfect information is that every information set is a singleton, imperfect information, in contrast, means that there is at least one nonsingleton information set.

Almost all games in the economics literature are game of *perfect recall*: No player ever forgets any information he once knew, and all players know the actions they have chosen previously. Figure 3.5 depicts a setting of imperfect recall. In this example, Player 1 first chooses between U and D and then chooses between X and Y. That nodes a and b are contained in the same information set means that Player 1 has forgotten whether he chose U or D when he makes the second choice.

Finally let us consider a game where a player can choose from an infinite number of actions.

Example 3.3. *Consider a simple market game between two firms in which firm 1 first decides how much to spend on advertising and then firm 2, after observing firm 1's choice, decides whether to exit or stay in the market. Suppose that firm 1 may choose any advertising level between zero and one (million dollars).*

Now firm 1 may choose advertising levels of .5, .3871, and so forth. Because there are an infinite number of possible actions for firm 1, we cannot draw a branch for each action.

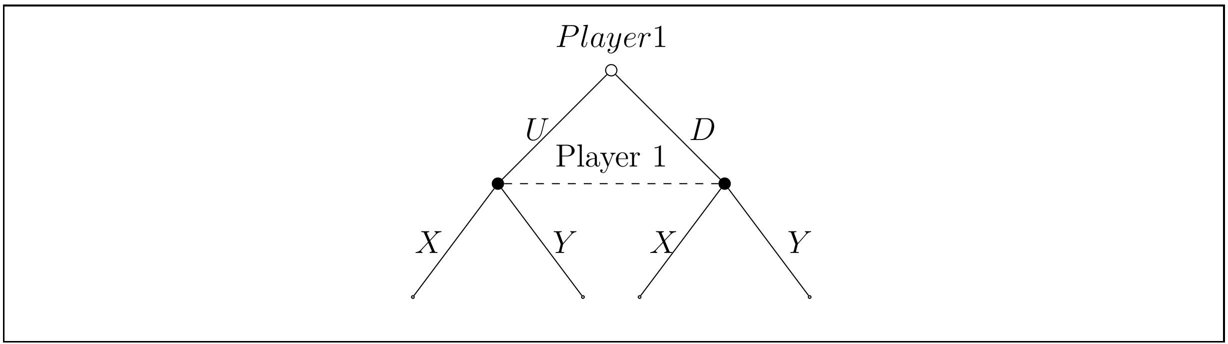


Figure 3.5: Imperfect Recall

One way of representing firm 1's potential choices, though, is to draw two branches from firm 1's decision node, designating advertising levels of zero and one, as in Figure 3.6(i).

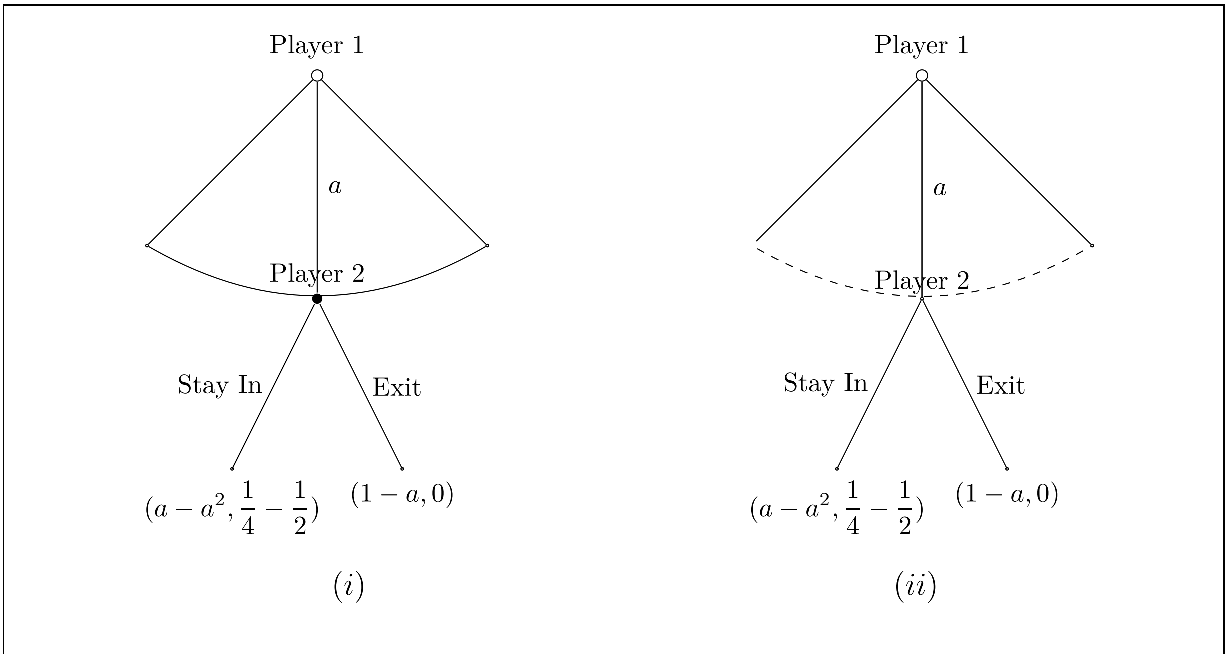


Figure 3.6: Advertising/ Exit

These branches are connected by an arc, which shows that firm 1 can select any number between zero and one. We label this graphical configuration with a variable (in this case, the letter a), which stands for firm 1's action. In the interior of the arc, we draw a generic node that continues the tree. In Figure 3.6 (i) Player 2 can observe the advertising level of Firm 1 before choosing his action. In another version of the game, Player 2 does not observe the advertising level of firm 1, as depicted in Figure 3.6 (ii).

3.1.2 Strategy

A strategy for a player in an extensive game must specify what he will do at each of his decision nodes. Hence, you can imagine a player telling a computer to play on his behalf. In that case, he does not know ex-ante which decision nodes will be reached. So, he gives the computer a complete contingent plan of what actions must be taken at every decision vertex.

Definition 3.2. *A* **strategy** *for a player in the game is a complete contingent plan of actions at every decision node.*

Let us now consider some examples.

Example 3.4. *Suppose there are two firms. One of them (say firm 1) decides whether to be aggressive in the market* (A)*, to be passive in the market* (P)*, or to leave the market* (O)*. If firm 1 leaves, then the other firm (firm 2) enjoys a monopoly. Otherwise the firms compete and firm 2 selects whether or not to assume an aggressive stance. Furthermore, when firm 2 makes its decision, it knows only whether firm 1 is in or out of the market; firm 2 does not observe firm 1's competitive stance before taking its action.*

In this game, there is one information set for firm 1 (the initial node) and one for firm 2. The strategy sets are $S_1 = \{A, P, O\}$ and $S_2 = \{A, P\}$. The game is drawn in Figure 3.7.

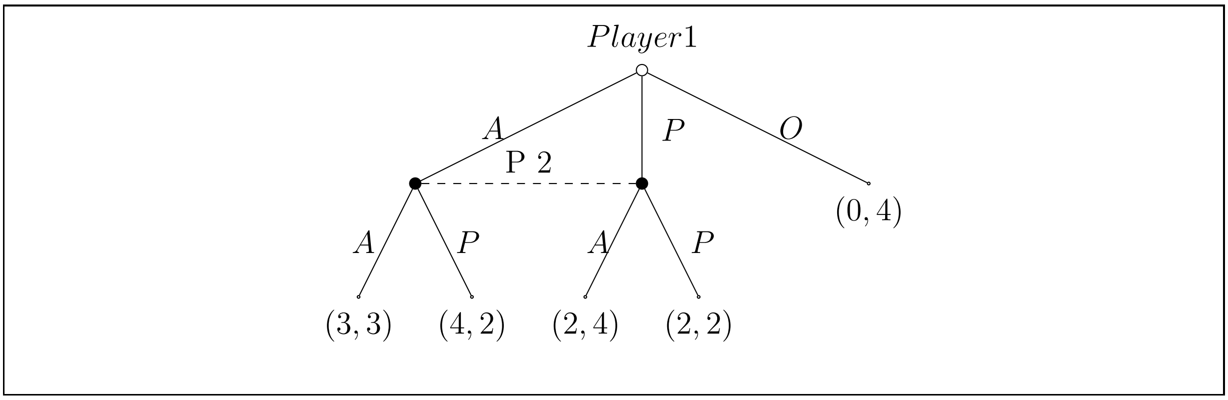


Figure 3.7: Marketing Strategy Game

Example 3.5. *Player 1 decides between "out"* (O) *and "in"* (I)*. If he chooses* O*, then the game ends with a payoff vector of* $(2, 2)$*. If he selects* I*, then player 2 is faced with the same two choices. If player 2 then chooses* O*, the game ends with the payoff vector* $(1, 3)$*. If she picks* I*, then player 1 has another choice to make, between* A *and* B *(ending the game with payoffs* $(4, 2)$ *and* $(3, 4)$*, respectively).*

Player 1 has two information sets, and player 2 has one information set. Note that in this game, player l's strategy must specify what he will do at the beginning of the game (O or I) and what action he would take at his second information set (A or B). There are four combinations of the two actions at each information set, and so there are four different strategies for player 1: $S_1 = \{OA, OB, IA, IB\}$.

At this point, you can see that "complete contingent plan" means more than just a "plan" for how to play the game. Indeed, why, you might wonder, should player 1 in

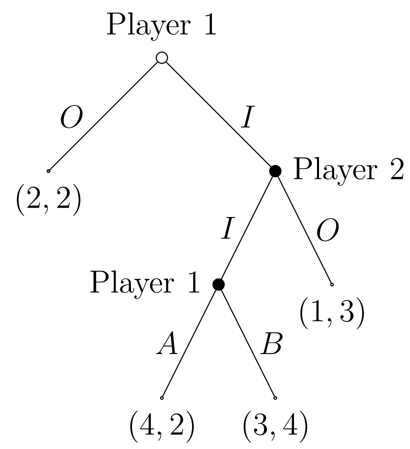


Figure 3.8: In and Out Game

the game shown in Figure 3.8 need a plan for his second information set if he selects O at the first? Can't we just say that player 1 has three different strategies, O, IA, and IB? In fact, no. The definition of a strategy (a complete contingent plan) requires a specification of player l's choice at his second information set even in the situation in which he plans to select O at his first information set. Furthermore, we will need to keep track of behavior at all information sets – even those that would be unreached if players follow their strategies – to fully analyze any game. That is, stating that player l's plan is O does not provide enough information for us to conduct a thorough analysis. The reason for this is that our study of rationality will explicitly require the evaluation of players' optimal moves starting from arbitrary points in a game. This evaluation is connected to the beliefs that players have about each other. For example, in the game depicted in Figure 3.8, player l's optimal choice at his first information set depends on what he thinks player 2 would do if put on the move. Furthermore, to select the best course of action, perspicacious player 2 must consider what player 1 would do at his second information set. Thus, player 2 must form a belief about player l's action at the third node. A belief is a conjecture about what strategy the other player is using; therefore, player l's strategy must include a prescription for his second information set, regardless of what this strategy prescribes for his first information set.

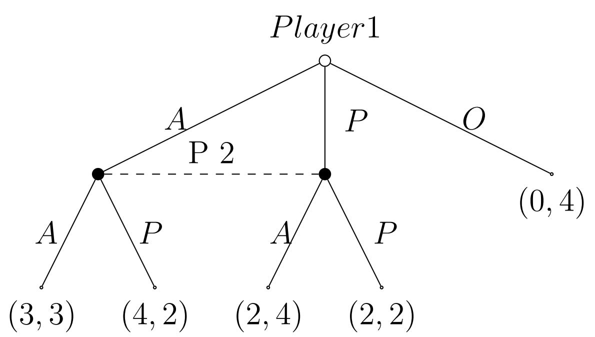
Example 3.6. *Consider a setting in which two players have to decide how much to contribute to a charitable project. Suppose it is known that the project will be successful if and only if the sum of the players' contributions is at least* \$900. *Further, the players know that they each have* \$600 *available to contribute. The players make contributions sequentially. In the first stage, player 1 selects her contribution* $x \in [0, 600]$*, and player 2 observes this choice. Then, in the second stage, player 2 selects his contribution* $y \in [0, 600]$*. If the project is successful, player l's payoff is* $800 - x$ *and player 2's payoff is* $800 - y$*.*

Here, 800 is the benefit each player derives from the successful project. If the project is unsuccessful, then player 1 gets $-x$ and player 2 gets $-y$. Note how many information sets each player has in this contribution game. Player 1 has just one information set (the initial node), and this player's strategy is simply a number between 0 and 600. Thus, $S_1 = [0, 600]$. Player 2, on the other hand, has an infinite number of information sets, each associated with a different value of x selected by player 1. At each of these information sets (each one a singleton node), player 2 must choose a contribution level. For instance, player 2's strategy must describe the amount he would contribute conditional on player 1 choosing $x = 500$, the amount he would contribute conditional on $x = 400$, and so on – for every value of x between 0 and 600. There is a simple mathematical way of expressing player 2's strategy. It is a function $s_2 : [0, 600] \to [0, 600]$, such that, for any given x, player 2 selects $y = s_2(x)$.

3.1.3 The Strategic-Form Representation of Extensive-Form Game

Our next step is to relate extensive-form game to the strategic-form model. For any game in extensive form, we can describe the strategy spaces of the players. Furthermore, notice that each strategy profile fully describes how the game is played. That is, a strategy profile tells us exactly what path through the tree is followed and, equivalently, which terminal node is reached to end the game. Associated with each terminal node (which we may call an outcome) is a payoff vector for the players. Therefore, each strategy profile implies a specific payoff vector. This is called the **reduced normal/strategic form** of the extensive game.

For example the game in Example 3.4 can be represented in normal form as follows:



Player 2

Player 1	A	P
A	$3, 3$	$4, 2$
P	$2, 4$	$2, 2$
O	$0, 4$	$0, 4$

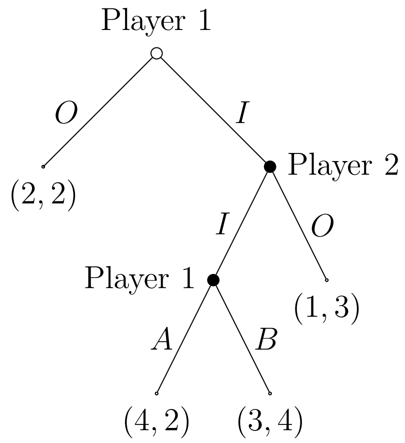
Figure 3.9: Marketing Strategy Game in Extensive and Noral Form

Similarly the game in example 3.8 can be represented in normal form as follows.

However, there can be several extensive forms with the same strategic form representation. See Figure 3.11 for example, both extensive forms yield the same normal form

.

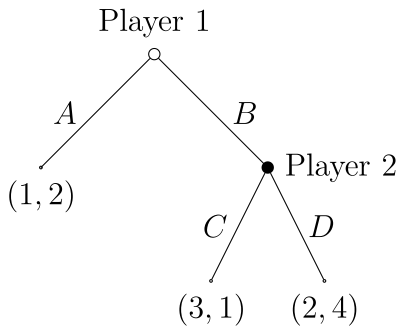
On the other hand, every strategic form game can be represented as an extensive form



Player 2

Player 1		I	O
	OA	2, 2	2, 2
	OB	2, 2	2, 2
	IA	4, 2	1, 3
	IB	3, 4	1, 3

Figure 3.10: In and Out Game in Extensive and Normal Form



Player 2

P 1		I	O
	A	1, 2	1, 2
	B	3, 1	2, 4

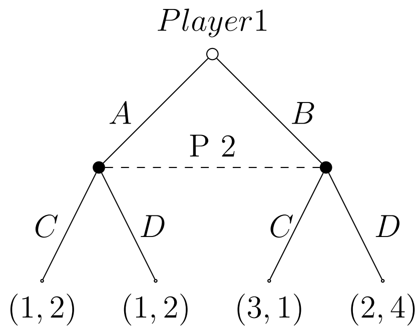


Figure 3.11: Corresponding Extensive and Normal Forms

game of imperfect information. To see this consider a strategic form game of two players: $N = \{1, 2\}$. In the strategic form game, each player $i \in N$ chooses an action from his strategy set S_i simultaneously. So, think of an extensive form game, where one of the players, say 1, moves first and chooses from $S_1 = \{l, m, r\}$. However, the action of Player 1 is not observed by Player 2 while he chooses a strategy from $S_2 = \{T, B\}$. Then, the game is shown in Figure 3.12. Notice that when Player 2 takes her action in Figure 3.12, he does not know which decision node he is in - so his three decision nodes are bundled in one information set.

We discuss equilibrium concepts for extensive form games, we start with extensive form games with perfect information.

3.2 Equilibria For Games of Perfect Information

Consider the entry game in example 3.1. One naive way of finding an equilibrium is to represent the game in (reduced) normal/ strategic form, and then apply the solution concepts of strategic form games. The reduced strategic form representation of this game

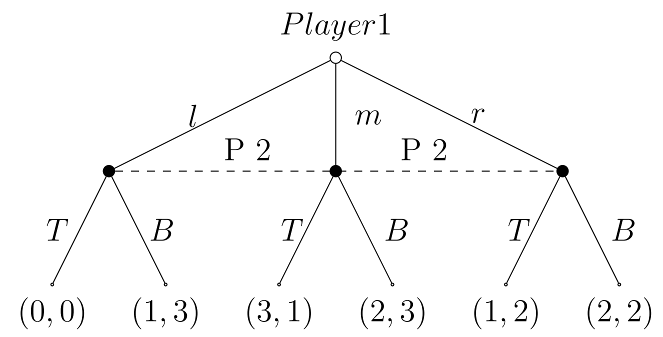
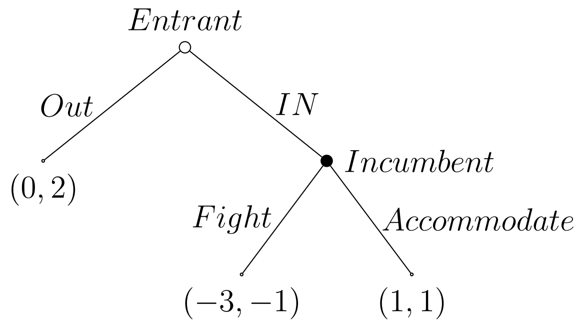


Figure 3.12: Strategic Form Game as an Extensive Form Game



		Incumbent	
		Fight	Accommodate
Entrant	Out	$\underline{0}, \underline{2}$	$0, \underline{2}$
	In	$-3, -1$	$\underline{1}, \underline{1}$

Figure 3.13: The Entry Game in Extensive and Reduced Strategic Form

is shown in Figure 3.13.

Observe in the game has two pure strategy Nash equilibria: $(Out, Fight)$ and $In, Accommodate)$. But note that once the game has reached the information set of Incumbent, he will play *Accommodate*. So, playing $Fight$ is not *credible* for Incumbent. Then, Entrant can take this information into account while choosing his action. Entrant clearly prefers playing In over Out since Incumbent cannot threaten him credibly to play $Fight$. Hence, the equilibrium $(Out, Fight)$ is not a good prediction of the game. The main idea here is that the equilibrium $(Out, Fight)$ specifies a strategy $Fight$ for Incumbent which is not a credible strategy – once the decision node of Incumbent is reached, he will never play this.

As we discussed above, a strategy profile leads to a unique terminal node with a unique path from the initial node to the terminal node. Hence, an equilibrium strategy profile will not touch on many decision nodes – these are called *off-equilibrium path* decision nodes. One primary requirement in extensive form game equilibrium is that action of every player must be optimal starting at every decision node, and not just decision nodes reached on equilibrium path.

We now look at *equilibrium refinements* which are designed to separate the "reasonable" Nash equilibrium from the "unreasonable" ones. In particular, we will discuss the ideas of *backward induction* and *subgame perfection*.

3.2.1 Subgame Perfect Equilibrium

This is the single most important solution concept for extensive form games. It enforces and formalizes the idea of credibility by using the notion of subgames.

Definition 3.3. *A* **subgame** *is an extensive-form game*

(a) begins at a decision node n that is a singleton information set

(b) includes all the decision and terminal nodes following n in the game tree (but no nodes that do not follow n), and

(c) does not cut any information sets (i.e., if a decision node n' follows n in the game tree, then all the other nodes in the information set containing n' must also follow n, and so must be included in the subgame).

Note that a game is a subgame of itself. So, every game has a subgame. Game in Figure 3.1 has two subgame: One starting with the Entrant's decision node and the other starting with Incumbent's decision node. The game in Figure 3.2 has only one one subgame which the game itself.

Definition 3.4. *(Selten 1965) A strategy profile s is a subgame perfect equilibrium (SPE) of the extensive form game if for every subgame of the game the strategy profile s restricted to that subgame is a Nash equilibrium of the subgame.*

Since a game itself is a subgame of the game, it follows that every SPE is a Nash equilibrium. We document this as a fact below.

Fact 3.1. *Every subgame perfect equilibrium is a Nash equilibrium.*

3.2.2 Backward Induction

An easy way to compute SPE in games with perfect information is the following. Start with a decision node just before a terminal node. Specify an action that leads to the highest payoff for the decision maker of that node among all possible actions – in case of ties, all possible actions leading to highest payoff are specified. If such an optimal action leads to terminal node z, then replace this decision node and the subsequent subgame by terminal node z. Repeat this procedure. If indifferences occur, this will lead to multiple strategy profiles surviving. This procedure is called the *backward induction procedure.*

Definition 3.5. *A strategy profile that survives the above procedure is said to be a strategy profile surviving the* **backward induction** *procedure.*

The following claim helps us to find SPE in a game with perfect information.

Claim 3.1. *A strategy profile is a subgame perfect equilibrium if and only if it survives the backwards induction procedure.*

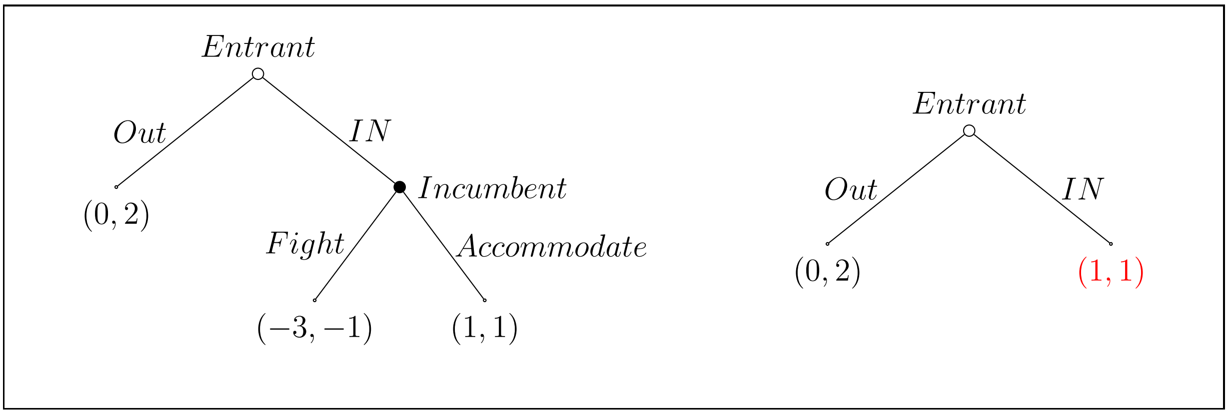


Figure 3.14: Subgame Perfect Equilibrium of The Entry Game

In the game in Figure 3.13, Incumbent plays *Accommodate*. Then we replace the subgame starting at the decision note of Entrant by payoff $(1, 1)$. Now, Entrant chooses In in this new game. Hence, the unique outcome of the backward induction procedure is $(In, Accommodate)$. The unique subgame perfect equilibrium (SPE) of this game is $(In, Accommodate)$.

3.2.3 Alternative Offers Bargaining

Example 3.7. Alternative Offers Bargaining *Two players are bargaining over 1 unit of money. They will bargain for T periods starting from period 0. In even periods, that is at $t = \{0, 2, ...\}$, Player 1 offers a split $(o_t, 1 - o_t)$, where $o_t \in [0, 1]$ is Player 1's share. If Player 2 accepts, the game ends. Else, we move to the next period. In odd periods, that is at $t = \{1, 3, ...\}$, Player 2 offers a split $(o_t, 1 - o_t)$, where $o_t \in [0, 1]$ is Player 1's share. Player 2 then either accepts or rejects. If no split is accepted at the end of period T, then the game ends with each player getting 0. Money received in period t is discounted by δ^t, where $\delta \in (0, 1)$.*

This game has perfect information, finite number of stages, but infinite set of actions at each decision node. There are many tied utilities too. But surprisingly, it has a unique subgame perfect equilibrium. To understand the game better, consider just a one-period, that is $T = 1$ case. This is also known as *ultimatum game*.

Ultimatum Game Player 1 offers a split $(o_0, 1 - o_0)$ and Player 2 can either accept or reject. We draw the game tree in Figure 3.15.

In all the decision nodes, where Player 2 gets a positive offer, he accepts. In the decision node where Player 2 gets zero offer, he is indifferent. Knowing this, we now apply backward induction on Player 1. Player 2 accepts an offer if

$$1 - o_0 \geq 0.$$

So, the unique subgame perfect equilibrium is $(1, 0)$.

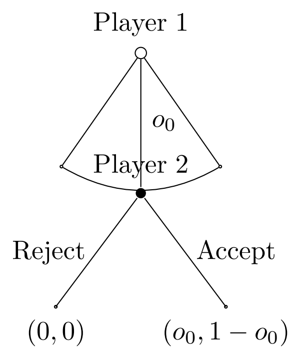


Figure 3.15: Ultimatum Game

Two-Period Bargaining Now suppose $T = 2$. At $t = 0$, Player 1 offers $(o_0, 1 - o_0)$. Player 2 either accepts or rejects the offer. If the offer is accepted, Player 1 gets o_0 and Player 2 gets $1 - o_0$ and the game ends. If Player rejects the offer, he makes a counter offer $(o_1, 1 - o_1)$. Player 1 either accepts or rejects. If he accepts, he gets o_1, present value of which is δo_1 and Player 2 gets $1 - o_1$, present value of which is $\delta(1 - o_1)$. We draw the game tree in Figure 3.16(i).

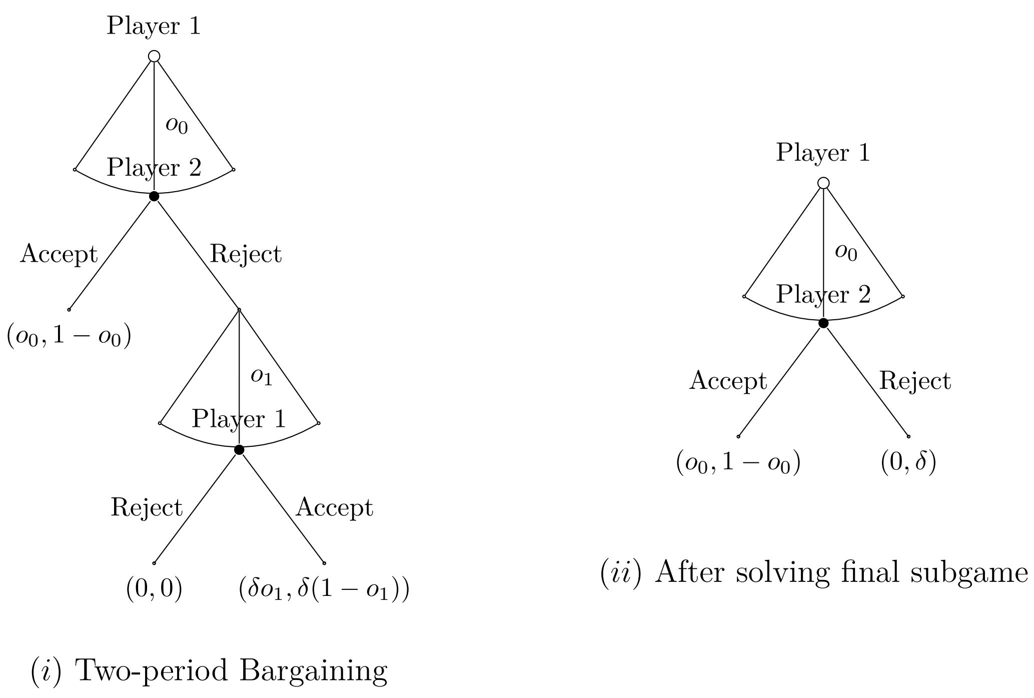


Figure 3.16: Two-period Bargaining and Solving Subgame Perfect Equilibrium

Now solve this using backward induction. In the last subgame, Player 1 accepts Player 2's offer if

$$\delta o_1 \geq 0.$$

So, Player 2 at $t = 1$ offers $(0, 1)$ and Player 1 accepts that (see Figure 3.16(ii).

Hence, at $t = 0$, Player 2 accepts Player 1's offer if

$$1 - o_0 \geq \delta.$$

So, Player 1 at $t = 0$ offers $(1 - \delta, \delta)$ and Player 2 accepts that. The unique SPE thus is $(1 - \delta, \delta)$.

Three-Period Bargaining Similarly we can solve for SPE when $T = 3$ by backward induction. Consider the final subgame, Player 2 accepts Player 1's offer $(o_2, 1 - o_2)$ if

$$\delta^2(1 - o_2) \geq 0.$$

So, here Player 1 offers $(1, 0)$ and Player 2 accepts that; Player 1's payoff is δ^2 and that of Player 2 is 0.

Now consider the subgame starting at $t = 1$. Player 1 thus accepts Player 2's offer $(o_1, 1 - o_1)$ if

$$\delta o_1 \geq \delta^2.$$

Hence, Player 2 offers $(\delta, 1 - \delta)$ and Player 1 accepts that. Player 1's payoff is δ and Player 2's is $1 - \delta$.

Finally, at $t = 0$, Player 2 accepts Player 1's offer $(o_0, 1 - o_0)$ if

$$1 - o_0 \geq \delta(1 - \delta).$$

Hence, Player 1 offers $(1 - \delta(1 - \delta), \delta(1 - \delta))$ and Player 2 accepts that. The unique SPE thus is $(1 - \delta(1 - \delta), \delta(1 - \delta))$.

Now we consider any arbitrary T, that is Player 1 and Player 2 bargain alternatively for T periods. We are interested in finding out subgame perfect equilibrium of the game.

First, suppose T is odd. Then, in the last period, Player 1 offers. Consider the subgame from this period. It consists of a decision node for Player 1 where he offers a split $(o_T, 1 - o_T)$ and a decision node for Player 2 for each offer of Player 1. Player 2 accepts if

$$1 - o_T \geq 0.$$

Thus, offering 0 and accepting 0 is the unique subgame perfect equilibrium outcome from period T. We now repeat this idea. Essentially, at each subgame an offer must be made such that the opponent is indifferent between accepting and rejecting and the opponent must accept. By backward induction, we proceed as follows.

1. In period T, Player 1 offers $(1, 0)$, which Player 2 accepts. Resulting payoffs are $(\delta^{T-1}, 0)$.

2. In period $(T-1)$, Player 1 can assure himself of δ^{T-1}. So, he accepts any offer giving him at least δ^{T-1}. So, Player 2 offers $(\delta, 1-\delta)$ which gives payoff $(\delta^{T-1}, \delta^{T-2} - \delta^{T-1})$.

3. In period $(T-2)$, Player 2 can assure himself of $\delta^{T-2}-\delta^{T-1}$. So, Player 1 offers $(1-\delta+\delta^2, \delta-\delta^2)$, which gives payoff $(\delta^{T-3}-\delta^{T-2}+\delta^{T-1}, \delta^{T-2}-\delta^{T-1})$.

   Continuing in this manner, we get

4. In period 0, Player 1 offers $(1-\delta+\delta^2-...+\delta^{T-1}, \delta-\delta^2+...-\delta^{T-1}) \equiv (\frac{1+\delta^T}{1+\delta}, \frac{\delta-\delta^T}{1+\delta})$, which is accepted by Player 2.

Now suppose T is even, a similar analysis yields an offer by Player 1 equal to $(\frac{1-\delta^T}{1+\delta}, \frac{\delta+\delta^T}{1+\delta})$
We summarize these in the following table

		Offerer	Receiver
One Stage $(T=1)$		1	0
Two Stage $(T=2)$		$1-\delta$	δ
Three Stage $(T=3)$		$1-\delta(1-\delta)$	$\delta(1-\delta)$
$\vdots$		$\vdots$	$\vdots$
Arbitrary T Stage	Odd T	$\frac{1+\delta^T}{1+\delta}$	$\frac{\delta-\delta^T}{1+\delta}$
	Even T	$\frac{1-\delta^T}{1+\delta}$	$\frac{\delta+\delta^T}{1+\delta}$

Figure 3.17: Alternative Offers Bargaining

Now suppose T be ∞. Observe, $\lim_{T\to\infty} \frac{1+\delta^T}{1+\delta} = \lim_{T\to\infty} \frac{1-\delta^T}{1+\delta} = \frac{1}{1+\delta}$ and $\lim_{T\to\infty} \frac{\delta-\delta^T}{1+\delta} = \lim_{T\to\infty} \frac{\delta+\delta^T}{1+\delta} = \frac{\delta}{1+\delta}$. Hence, $T \to \infty$, the equilibrium converge to $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$. At $\delta = 1$ the equilibrium payoffs are $(\frac{1}{2}, \frac{1}{2})$.

Observation 3.1. *When the players are patient (that is do not discount future at all $\delta = 1$), and they can potentially bargain forever, at equilibrium they equally share the money.*

Finally note that in all the cases, first offer is accepted – the players know the game, so they know the equilibrium strategies, the offerer offers that amount at $t = 0$ and the receiver immediately accepts that.

3.2.4 Stackelberg Model of Duopoly

Example 3.8. Stackelberg Model of Duopoly *In a market there are two firms, both producing the same good. Let marginal cost be c and the demand function be $p(q) = a - bq$.*

Like in Cournot game, the firms choose quantities, but the firms make their decisions sequentially, rather than simultaneously: One firm chooses its output, then the other firm does so, knowing the output chosen by the first firm.

The timing of the game is as follows: (i) Firm 1 chooses a quantity $q_1 \geq 0$; (ii) Firm 2 observes that q_1 and then chooses a quantity $q_2 \geq 0$; (iii) The payoff to Firm i is given by the profit function

$$\pi_i(q_i, q_j) = q_i[p(q) - c],$$

where $p(q) = a - bq$ is the market clearing price when the aggregate quantity of the market is $q = q_1 + q_2$, and c is the constant marginal cost of production (fixed costs being zero). We draw the game tree in Figure 3.18.

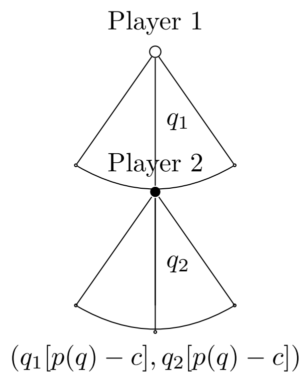


Figure 3.18: Stackelberg Game

To solve for the backward-induction outcome of this game, we first compute Firm 2's reaction to an arbitrary quantity of Firm 1. Given Firm 1's quantity q_1, the problem of Firm 2 is to

$$\max_{q_2 \geq 0} \pi_2(q_1, q_2) = \max_{q_2 \geq 0} q_2[a - b(q_1 + q_2) - c],$$

which yields

$$BR_2(q_1) = \frac{a - c}{2b} - \frac{q_1}{2},$$

provided $q_1 < \dfrac{a - c}{b}$. The same equation for $BR_2(q_1)$ appeared in our analysis of the simultaneous-move Cournot game in Section 2.4.1. The difference is that here $BR_2(q_1)$ is truly Firm 2's reaction to Firm 1's *observed* quantity, whereas in the Cournot analysis $BR_2(q_1)$ is Firm 2's best response to a *hypothesized* quantity to be simultaneously chosen by Firm 1.

Now since Firm 1 can solve Firm 2's problem, Firm 1 will anticipate that the quantity choice q_1 will be met with the reaction $BR_2(q_1)$. Thus, Firm 1's problem in the first stage of the game is to

$$\max_{q_1 \geq 0} \pi_1(q_1, BR_2(q_1)) = \max_{q_1 \geq 0} q_1[a - b(q_1 + BR_2(q_1)) - c] = \max_{q_1 \geq 0} q_1[\frac{a - c}{2} - \frac{bq_1}{2}]$$

which yields

$$q_1^S = \frac{a-c}{2b} \text{ and consequently } q_2^S = BR_2(q_1^S) = \frac{a-c}{4b}$$

as the backward induction outcome of the Stackelberg duopoly game.

The aggregate quantity is $q^S = q_1^S + q_2^S = \dfrac{3(a-c)}{4b}$ and $p^S = \dfrac{a+3c}{4}$.

Recall in Cournot game we found that each firm produces $\dfrac{a-c}{3b}$, hence the aggregate quantity is $\dfrac{2(a-c)}{3b}$ and price is $\dfrac{a+2c}{3}$. Thus, aggregate quantity in the backward induction outcome of the Stackelberg game is lower than the aggregate quantity in the Nash equilibrium of the Cournot game. Hence, the market clearing price is lower in the Stackelberg game.

Finally, profit of Firm 1 is $\pi_1^S = \Big[\dfrac{a+c}{4} - c\Big]\dfrac{a-c}{2b} = \dfrac{(a-c)^2}{8b}$ and that of Firm 2 is $\pi_2^S = \Big[\dfrac{a+c}{4} - c\Big]\dfrac{a-c}{4b} = \dfrac{(a-c)^2}{16b}$.

Recall in Cournot game we found that profit of each firm is $\pi_i^O = \Big[\dfrac{a+2c}{3} - c\Big]\dfrac{a-c}{3b} = \dfrac{(a-c)^2}{9b}$. Hence,

$$\pi_2^S < \pi_2^O = \pi_1^O = \pi_1^S.$$

Intuitively, in the Stackelberg game, Firm 1 could have chosen its Cournot quantity $\dfrac{a-c}{3b}$, in which case Firm 2 would have responded with its Cournot quantity. Thus, in the Stackelberg game, Firm 1 could have achieved its Cournot profit level but chose to do otherwise, so Firm 1's profit in the Stackelberg game must exceed its profit in the Cournot game (Check this). But the market clearing price is lower in the Stackelberg game, so aggregate profits are lower, so the fact that Firm 1 is better off implies that Firm 2 is worse-off in the Stackelberg than in the Cournot game.

The observation that Firm 2 does worse in Stackelbderg than in the Cournot game illustrates an important *difference between single and multi person* decision problems. In the single-person decision theory, having more information can never make the decision maker worse off. In game theory, however, having more information (or, more precisely, having it known to the other players that one has more information) *can*[1] make a player worse off.

In the Stackelberg game, the information in question is Firm 1's quantity: FIrm 2 knows q_1, and (as importantly) Firm 1 knows that Firm 2 knows q_1. To see the effect this information has, consider the modified sequential-move game in which Firm 1 chooses q_1. If Firm 2 believes that Firm 1 has chosen its Stackelberg quantity $q_1^S = \dfrac{a-c}{2b}$, then Firm 2's best-response is again to choose $q_2^S = BR_2(q_1^S) = \dfrac{a-c}{4b}$. *But* if Firm 1 anticipates that Firm 2 will hold this belief and so choose this quantity, then Firm

[1] Note the word "can".

1 prefers to choose its best response to $\frac{a-c}{4b}$, which is $q_1 = BR_1(\frac{a-c}{4b}) = \frac{a-c}{2b} - \frac{a-c}{8b} = \frac{3(a-c)}{8b}$ (check this quantity indeed provides Firm 1 higher profit than what does the Stackelberg quantity, given that Firm 2 chooses to produce $\frac{a-c}{4b}$), rather than its Stackelberg quantity. Thus, Firm 2 should not believe that Firm 1 has chosen q_1^S. Rather, the unique Nash equilibrium of this modified sequential-move game is for both firms to choose the quantity $\frac{a-c}{3b}$, precisely the Nash equilibrium of the Cournot game, where the firms moves simultaneously.[2] Thus, having Firm 1 know that Firm 2 knows q_1 hurts Firm 2.

> The idea that backward induction gives the right answer in simple games was implicit in the economics literature before Selten's paper. In particular, it is embodied in the idea of Stackelberg equilibrium: The requirement that Player 2's strategy be the Cournot reaction function is exactly the idea of backward induction.

Backward induction can be a very demanding solution in games where players need to move many times. This is because it requires players to anticipate actions down the game tree. A sharp example of this fact is given a well known game called the **centipede game.**

Example 3.9. Centipede Game *Two players start with 1 unit of money each. Each player can either decide to continue C or stop S. If anyone stops, then the game ends and each take their piles. If a player continues, then the opponent gets to take action but his pile is reduced by* 1 *while the opponent's pile is increased by* 2*. The play ends when both the players reach* 100*.*

Suppose Player 1 moves first. Unique prediction due to backward induction is Player 1 stops in the first chance resulting in $(1, 1)$. The subgame perfect equilibrium specifies action S at every decision vertex. This is also the unique Nash equilibrium of this game.

In lab experiments, agents have usually continued for some time. This is a general critique of equilibrium in extensive form game that no satisfactory refinement can predict such an outcome. We will often refer to all these notions to be the definition of a subgame perfect equilibrium in such games.

A subgame perfect equilibrium in pure strategies always exist – this follows from the fact that the backward induction procedure always generates at least one pure strategy profile. If there are no indifferences in payoffs, the backward induction procedure generates a unique strategy profile, which is referred to as the backward induction solution.

Now we discuss equilibrium concepts for extensive form games with imperfect information.

[2] This emphasizes the fact that what matters is information and not time.

3.3 Equilibria For Games of Imperfect Information

In games where there is imperfect information, subgame perfect equilibrium can still be applied but backward induction is not well-defined in such games. Moreover, subgame perfect equilibrium may be a useless solution concept in which there is imperfect information. To see this, consider the game in example 3.4. This game has only one subgame. Hence, the set of Nash equilibria are equivalent to the set of subgame perfect equilibria. The problem with subgame perfect equilibrium in this game is that it does not use any *beliefs* of Player 2. As a result, it puts no restriction on his optimal choice when his information set is reached. To appropriately define behavior in information sets, any equilibrium must also define beliefs and equilibrium choices must be consistent with these beliefs. This is the basic idea behind defining equilibrium refinements in games of imperfect information. We will come back to this solution concept later in our course.

We end this chapter with *repeated game* – one of the best understood class of dynamic games.

3.4 Repeated Games

In chapter 2, we dealt with one-stage games, which model situations where the interaction between the players takes place only once, and once completed, it has no effect on future interactions between the players. In many cases, interaction between players does not end after only one encounter; players often meet each other many times, either playing the same game over and over again, or playing different games. There are many examples of situations that can be modeled as multistage interactions: a printing office buys paper from a paper manufacturer every quarter; a tennis player buys a pair of tennis shoes from a shop in his town every time his old ones wear out; baseball teams play each other several times every season. In this section we will study *repeated games*. A repeated game consists of a base game, which is a game in strategic form, that is repeated either finitely or infinitely many times.

These games are of particular interest because when players repeatedly encounter each other in strategic situations, behavioral phenomena emerge that are not present in one-stage games.

- The very fact that the players encounter each other repeatedly gives them an opportunity to cooperate, by conditioning their actions in every stage on what happened in previous stages. A player can threaten his opponent with the threat "if you do not cooperate now, in the future I will take actions that harm you," and he can carry out this threat, thus "punishing" his opponent. For example, the manager of a printing office can inform a paper manufacturer that if the price of the paper he purchases is not reduced by 10% in the future, he will no longer buy paper from that manufacturer.

- Repeated games enable players to develop *reputations*. A sporting goods shop can develop a reputation as a quality shop, or a discount store. In this chapter, we present the model of repeated games. This is a simple model of games in which players play the same base game time and again. In particular, the set of players, the actions available to the players, and their payoff functions do not change over time, and are independent of past actions. This assumption is, of course, highly restrictive, and it is often unrealistic: in the example above, new paper manufacturers enter the market, existing manufacturers leave the market, there are periodic changes in the price of paper, and the quantity of paper that printers need changes over time. This simple model, however, enables us to understand some of the phenomena observed in multistage interactions. The more general model, where the actions of the players and their payoff functions may change from one stage to another, is called the model of "stochastic games" (which we will not cover in this course).

We start with two-stage repeated games and study finitely repeated games where the *stage game* is played for T times and T is finite. Then we study infinitely repeated games.

3.4.1 Finitely Repeated Games

We start with two-stage repeated game – consider the Prisoner's Dilemma given in normal form in Figure 3.19.

		Player 2	
		L_2	R_2
Player 1	L_1	$1,1$	$5,0$
	R_1	$0,5$	$4,4$

Figure 3.19: Prisoner's Dilemma: Two-Stage Repeated Game

Suppose two players play this simultaneous-move game twice, observing the outcome of the first play before the second play begins, and suppose the payoff for the entire game is simply the sum of the payoffs from the two stages (i.e., there is no discounting). We will call this repeated game the two-stage Prisoner's Dilemma.

We can think of reduced strategic form of this game. In this reduced form, Player $i \in \{1,2\}$ has a complex strategy. First, she needs to choose an action for Stage 1. Second, she needs to choose an action for every observed action profile of Stage 1 for Stage 2. For instance, if she has observed, (L_1, R_2) being played in Stage 1, her Stage 2 choice of an action can be contingent on that. This leads to a very complex strategy structure of the game in reduced form. Instead of looking at the reduced form, we can also analyze the game backwards.

Observe, the unique equilibrium of the second-stage game is (L_1, L_2), regardless of the first-stage outcome. Now, we analyze the first stage of the two-stage Prisoner's Dilemma by taking into account that the outcome of the game remaining in the second stage will be the Nash equilibrium of that remaining game – namely, (L_1, L_2) with payoff $(1, 1)$.

		Player 2	
		L_2	R_2
Player 1	L_1	$2, 2$	$6, 1$
	R_1	$1, 6$	$5, 5$

Figure 3.20: Solving Prisoner's Dilemma: Two-Stage Repeated Game

Thus, the players' first-stage interaction in the two-stage Prisoner's Dilemma amounts to the one-shot game in Figure 3.20, in which the payoff pair $(1, 1)$ for the second stage has been added to each first-stage payoff pair. The game in Figure 3.20 also has a unique Nash equilibrium: (L_1, L_2). Thus, the unique subgame-perfect outcome of the two-stage Prisoner's Dilemma is (L_1, L_2) in the first stage, followed by (L_1, L_2) in the second stage. Cooperation – that is, (R_1, R_2) – cannot be achieved in either stage of the subgame-perfect outcome.

This argument holds more generally. (Here we temporarily depart from the two-period case to allow for any finite number of repetitions, T.) Let $G = (N, \{A_i\}_{i=}^N, \{u_i\}_{i=1}^N)$ denote a static game of complete information. The game G will be called the stage game of the repeated game.

Definition 3.6. *Given a stage game G, let $G(T)$ denote the* **finitely repeated game** *in which G is played T times with actions taken by of all players in the preceding stages observed before the play in the next stage, and payoffs of $G(T)$ are simply the sum of payoffs in all T stages.*

Our arguments earlier lead to the following proposition (without formally defining notions of equilibrium).

Proposition 3.1. *If the stage game G has a unique Nash equilibrium, then for any finite repetition of G, the repeated game $G(T)$ has a unique subgame perfect outcome: the Nash equilibrium of the stage game G is played in every stage.*

There are two important assumptions here:

(a) the stage game has a unique Nash equilibrium

(b) the stage game is repeated finite number of times.

We will see that if either of the two assumptions are not present then it is possible for players to get better payoffs.

We now return to the two-period case, but consider the possibility that the stage game G has multiple Nash equilibria, as in Figure 3.21.

Player 2

	L_2	M_2	R_2
Player 1 L_1	$1, 1$	$5, 0$	$0, 0$
M_1	$0, 5$	$4, 4$	$0, 0$
R_1	$0, 0$	$0, 0$	$3, 3$

Figure 3.21: Prisoner's Dilemma with Multiple Nash Equilibria: Two-Stage Repeated Game

The strategies labeled L_i and M_i, mimic the Prisoner's Dilemma from Figure 3.19, but the strategies labeled R_i, have been added to the game so that there are now two pure-strategy Nash equilibria. (L_1, L_2), as in the Prisoner's Dilemma, and now also (R_1, R_2). It is of course artificial to add an equilibrium to the Prisoner's Dilemma in this way, but our interest in this game is expositional rather than economic. In the next subsection we will see that infinitely repeated games share this multiple-equilibria spirit even if the stage game being repeated infinitely has a unique Nash equilibrium, as does the Prisoner's Dilemma. Thus, in this subsection we analyze an artificial stage game in the simple two-period framework, and thereby prepare for our later analysis of an economically interesting stage game in the infinite-horizon framework.

Suppose the stage game in Figure 3.21 is played twice, with the first-stage outcome observed before the second stage begins. We will show that there is a subgame-perfect outcome of this repeated game in which the strategy pair (M_1, M_2) is played in the first stage. Like before, assume that in the first stage the players anticipate that the second-stage outcome will be a Nash equilibrium of the stage game. Since this stage game has more than one Nash equilibrium, it is now possible for the players to anticipate that different first-stage outcomes will be followed by different stage-game equilibria in the second stage. Suppose, for example, that the players anticipate that (R_1, R_2) will be the second-stage outcome if the first-stage outcome is (M_1, M_2), but that (L_1, L_2) will be the second-stage outcome if any of the eight other first-stage outcomes occurs.

Player 2

	L_2	M_2	R_2
Player 1 L_1	$2, 2$	$6, 1$	$1, 1$
M_1	$1, 6$	$7, 7$	$1, 1$
R_1	$1, 1$	$1, 1$	$4, 4$

Figure 3.22: Two-Stage Repeated Game: Analyzing Payoffs of First Stage

This means, in the first stage of the game, the players are looking at a payoff table as in Figure 3.22, where second stage payoff $(3, 3)$ is added to (M_1, M_2) and second stage payoff $(1, 1)$ is added to the eight other cells. The addition of different payoffs to different strategy profiles changes the equilibria of this game.

There are three pure-strategy Nash equilibria in the game in Figure 3.22: (L_1, L_2), (M_1, M_2) and (R_1, R_2). As in Figure 3.20, Nash equilibria of this one-shot game correspond to subgameperfect outcomes of the original repeated game. The Nash equilibrium (L_1, L_2) in Figure 3.22 corresponds to the subgame-perfect outcome $(L_1, L_2), L_1, L_2))$ in the repeated game, because the anticipated second stage outcome is (L_1, L_2) following anything but (M_1, M_2) in the first stage. Likewise, the Nash equilibrium (R_1, R_2) in Figure 3.22 corresponds to the subgame-perfect outcome $(R_1, R_2), (L_1, L_2))$ in the repeated game. These two subgame-perfect outcomes of the repeated game simply concatenate Nash equilibrium outcomes from the stage game, but the third Nash equilibrium in Figure 3.22 yields a qualitatively different result: (M_1, M_2) in Figure 3.22 corresponds to the subgame-perfect outcome $((M_1, M_2), (R_1, R_2))$ in the repeated game, because the anticipated second-stage outcome is (R_1, R_2) following (M_1, M_2). Thus, as claimed earlier, cooperation can be achieved in the first stage of a subgame-perfect outcome of the repeated game.

This is an example of a more general point: if $G = (N, \{A_i\}_{i=}^N, \{u_i\}_{i=1}^N)$ is a static game of complete information with multiple Nash equilibria then there may be subgame-perfect outcomes of the repeated game $G(T)$ in which, for any $t < T$, the outcome in stage t is not a Nash equilibrium of G. We return to this idea in the infinite-horizon analysis in the next subsection.

The main point to extract from this example is that *credible threats* or promises about future behavior can influence current behavior. A second point, however, is that subgame-perfection may not embody a strong enough definition of credibility. In deriving the subgame-perfect outcome $((M_1, M_2), (R_1, R_2))$, for example, we assumed that the players anticipate that (R_1, R_2) will be the second-stage outcome if the first-stage outcome is (M_1, M_2) and that (L_1, L_2) will be the second-stage outcome if any of the eight other first-stage outcomes occurs.

But playing (L_1, L_2) in the second stage, with its payoff of $(1, 1)$, may seem silly when (R_1, R_2), with its payoff of $(3, 3)$, is also available as a Nash equilibrium of the remaining stage game. Loosely put, it would seem natural for the players to *renegotiate*. If (M_1, M_2) does not occur as the first-stage outcome, so that (L_1, L_2) is supposed to be played in the second stage, then each player might reason that bygones are bygones and that the unanimously preferred stage – game equilibrium (R_1, R_2) should be played instead. But if (R_1, R_2) is to be the second-stage outcome after *every* first-stage outcome, then the incentive to play (M_1, M_2) in the first stage is destroyed: The first-stage interaction between the two players simply amounts to the one-shot game in which the payoff $(3, 3)$ has been added to each cell of the stage game in Figure 3.21, so L_i, is player i's best response to M_i.

To suggest a solution to this renegotiation problem, we consider the game in Figure 3.23, which is even more artificial than the game in Figure 3.21. Once again, our interest in this game is expositional rather than economic. The ideas we develop here to address renegotiation in this artificial game can also be applied to renegotiation in infinitely repeated games.

Player 2

Player 1		L_2	M_2	R_2	P_2	Q_2
	L_1	$1,1$	$5,0$	$0,0$	$0,0$	$0,0$
	M_1	$0,5$	$4,4$	$0,0$	$0,0$	$0,0$
	R_1	$0,0$	$0,0$	$3,3$	$0,0$	$0,0$
	P_1	$0,0$	$0,0$	$0,0$	$4,0.5$	$0,0$
	Q_1	$0,0$	$0,0$	$0,0$	$0,0$	$0.5,4$

Figure 3.23: Two-Stage Repeated Game: Renegotiation Proofness

This stage game adds the strategies P_i and Q_i, to the stage game in Figure 3.21. There are four pure-strategy Nash equilibria of the stage game: (L_1, L_2), (R_1, R_2), (P_1, P_2) and (Q_1, Q_2). As before, the players unanimously prefer (R_1, R_2) to (L_1, L_2). More importantly, there is no Nash equilibrium (x, y) in Figure 3.23 such that the players unanimously prefer (x, y) to (P_1, P_2), or (Q_1, Q_2), or (R_1, R_2). We say that (R_1, R_2) *Pareto dominates* Pardodominates (L_1, L_2), and that (P_1, P_2), (Q_1, Q_2), and (R_1, R_2) are on the Pareto frontier of the payoffs to Nash equilibria of the stage game in Figure 3.23.

Suppose the stage game in Figure 3.23 is played twice, with the first-stage outcome observed before the second stage begins. Suppose further that the players anticipate that the second-stage outcome will be as follows:

- (R_1, R_2) if the first-stage outcome is (M_1, M_2)
- (P_1, P_2) if the first-stage outcome is (M_1, w), where w is anything but M_2
- (Q_1, Q_2) if the first-stage outcome is (x, M_2), where x is anything but M_1
- (R_1, R_2) if the first-stage outcome is (y, z), where y is anything but M_1 and z is anything but M_2.

Then $((M_1, M_2), (R_1, R_2))$ is a subgame-perfect outcome of the repeated game, because Player i gets $4 + 3$ from playing M_i, and then R_i, but only $5 + 0.5$ from deviating to L_i in the first stage (and even less from other deviations).

More importantly, the difficulty in the previous example does not arise here. In the two-stage repeated game based on Figure 3.21, the only way to punish a player for deviating in the first stage was to play a Pareto-dominated equilibrium in the second stage, thereby also punishing the punisher. Here, in contrast, there are three equilibria on the Pareto frontier – one to reward good behavior by both players in the first stage, and two others to be used not only to punish a player who deviates in the first stage but also to reward the punisher. Thus, if punishment is called for in the second stage, there is no other stage-game equilibrium the punisher would prefer, so the punisher cannot be persuaded to renegotiate the punishment.

3.4.2 Infinitely Repeated Games

We now turn to infinitely repeated games. As in the finite-horizon case, the main theme is that credible threats or promises about future behavior can influence current behavior. In the finite-horizon case we saw that if there are multiple Nash equilibria of the stage game G then there may be subgame-perfect outcomes of the repeated game $G(T)$ in which, for any $t < T$, the outcome of stage t is not a Nash equilibrium of G. A stronger result is true in infinitely repeated games: even if the stage game has a unique Nash equilibrium, there may be subgame-perfect outcomes of the infinitely repeated game in which no stage's outcome is a Nash equilibrium of G.

We begin by studying the infinitely repeated Prisoner's Dilemma. We then consider the class of infinitely repeated games analogous to the class of finitely repeated games defined in the previous section: a static game of complete information, G, is repeated infinitely, with the outcomes of all previous stages observed before the current stage begins. For these classes of finitely and infinitely repeated games, we define a player's strategy, a subgame, and a subgame-perfect Nash equilibrium. We then use these definitions to state the Folk Theorem.

Suppose the Prisoner's' Dilemma in Figure 3.19 is to be repeated infinitely and that, for each t, the outcomes of the $t-1$ preceding plays of the stage game are observed before the t^{th} stage begins. Simply summing the payoffs from this infinite sequence of stage games does not provide a useful measure of a player's payoff in the infinitely repeated game. In particular we assume that players discount future payoffs (recall in bargaining model also we did this). Let δ be the today's value of a dollar to be received one stage later.

> One interpretation is that $\delta = \dfrac{1}{1+r}$, where r in the interest rate. The intuition is if a person save δ dollar today he will get back $(1+r)\cdot\delta = 1$ dollar in the next stage, so the present value of that dollar is δ.
>
> We can also use δ to reinterpret what we call an infinitely repeated game as a repeated game that ends after a random number of repetitions. Suppose that after each stage is played a (weighted) coin is flipped to determine whether the game will end. If the probability is p that the game ends immediately, and therefore $1-p$ that the game continues for at least one more stage, then a payoff π to be received in the next stage *(if it is played)* is worth only $(1-p)\dfrac{\pi}{1+r}$ before this stage's coin flip occurs. Likewise, a payoff π to be received two stages from now (if both it and the intervening stage are played) is worth only $(1-p)^2\dfrac{\pi}{(1+r)^2}$ before this stage's coin flip occurs. Let $\delta = \dfrac{1-p}{1+r}$. Then the present value reflects both the time-value of money and the possibility that the game will end.

Definition 3.7. *Given the discount factor δ, the present value of the infinite sequence of payoffs π_0, π_1,π_2,... is*

$$\pi_0 + \delta\pi_1 + \delta^2\pi_2 + \ldots = \sum_{t=0}^{\infty} \delta^t \pi_t.$$

Consider the infinitely repeated Prisoner's Dilemma in which each player's discount factor being δ and each player's payoff in the repeated game is the present value of the player's payoffs from the stage games. We will show that cooperation – that is, (R_1, R_2) – can occur in every stage of a subgame-perfect outcome of the infinitely repeated game, even though the *only* Nash equilibrium in the stage game is noncooperation – that is, (L_1, L_2). The argument is in the spirit of our analysis of the two-stage repeated game based on Figure 3.19 (the stage game in which we added a second Nash equilibrium to the Prisoners' Dilemma): if the players cooperate today then they play a high-payoff equilibrium tomorrow; otherwise they play a low-payoff equilibrium tomorrow.

The *difference* between the two-stage repeated game and the infinitely repeated game is that here the high-payoff equilibrium that might be played tomorrow is not artificially added to the stage game but rather represents continuing to cooperate tomorrow and thereafter. Suppose player i begins the infinitely repeated game by cooperating and then cooperates in each subsequent stage game if and only if both players have cooperated in every previous stage. Formally, player i's strategy is:

> Play R_i in the first stage. In the t^{th} stage, if the outcome of all $t-1$ preceding stages has been (R_1, R_2) then play R_i , otherwise, play L_i.

This strategy is an example of a *trigger strategy*, so called because player i cooperates until someone fails to cooperate, which triggers a switch to noncooperation forever after. If both players adopt this trigger strategy then the outcome of the infinitely repeated game will be (R_1, R_2) in every stage. We first argue that if δ is close enough to one then it is a Nash equilibrium of the infinitely repeated game for both players to adopt this strategy. We then argue that such a Nash equilibrium is subgame-perfect.

To show that it is a Nash equilibrium of the infinitely repeated game for both players to adopt the trigger strategy, we will assume that player i has adopted the trigger strategy and then show that, provided δ is close enough to one, it is a best response for player j to adopt the strategy also. Since player i will play L_i, forever once one stage's outcome differs from (R_1, R_2), player j's best response is indeed to play L_j, forever once one stage's outcome differs from (R_1, R_2). It remains to determine player j's best response in the first stage, and in any stage such that all the preceding outcomes have been (R_1, R_2). PlayingL_j will yield a payoff of 5 this stage but will trigger noncooperation by player i (and therefore also by Player j) forever after, so the payoff in every future stage will be 1, the present value of this sequence of payoffs is

$$5 + \delta \cdot 1 + \delta^2 \cdot 1 + ... = 5 + \frac{\delta}{1-\delta}.$$

Alternatively, playing R_j, will yield a payoff of 4 in this stage and will lead to exactly the same choice between L_j and R_j, in the next stage.

Now let V denote the present value of the infinite sequence of payoffs player j receives from making this choice optimally (now and every time it arises subsequently). If playing

R_j, is optimal then

$$V = 4 + \delta V \quad \Rightarrow V = \frac{4}{1-\delta},$$

because playing R_j, leads to the same choice next stage. If playing L_j is optimal then

$$V = 5 + \frac{\delta}{1-\delta}$$

as derived earlier. So playing R_j, is optimal if and only if

$$\frac{4}{1-\delta} \geq 5 + \frac{\delta}{1-\delta} \quad \Rightarrow \delta \geq \frac{1}{4}.$$

Thus, in the first stage, and in any stage such that all the preceding outcomes have been (R_1, R_2), Player j's optimal action (given that Player i has adopted the trigger strategy) is R_j, if and only if $\delta \geq \frac{1}{4}$. Combining this observation with the fact that Player j's best response is to play L_j forever once one stage's outcome differs from (R_1, R_2), we have that it is a Nash equilibrium for both players to play the trigger strategy if and only if $\delta \geq \frac{1}{4}$.

We now want to argue that such a Nash equilibrium is subgame-perfect. To do so, we define a strategy in a repeated game. Recall in the previous subsection, we defined the finitely repeated game $G(T)$ based on a stage game G – a static game of complete information in which players 1 through N simultaneously choose actions a_1 through a_N from the action spaces A_1 through A_N, respectively, and payoffs are $u_1(a_1, ..., a_N)$ through $u_N(a_1, ..., a_N)$. We now define the analogous for infinitely repeated games.

Definition 3.8. *Given a stage game G, let $G(\infty, \delta)$ denote the infinitely repeated game in which G is repeated forever and the players share the discount factor δ. For each t, the outcomes of the $t-1$ preceding plays of the stage game are observed before the t^{th} stage begins. Each player's payoff in $G(\infty, \delta)$ is the present value of the player's payoffs from the infinite sequence of stage games.*

Now, a player's strategy is a complete contingent plan of action. So, we define *history* of a game first.

Definition 3.9. *In the finitely repeated game $G(T)$ or the infinitely repeated game $G(\infty, \delta)$, the history of play through stage t is the record of the players' choices in stages 1 through t. The players might have chosen $(a_{11}, ..., a_{N1})$ in stage 1, $(a_{12}, ..., a_{N2})$ in stage 2, $(a_{1t}, ..., a_{Nt})$ at any stage t, for example, where for each player i and stage t the action a_{it} belongs to the action space A_i.*

Now, we define strategy in this repeated game context.

Definition 3.10. *In the finitely repeated game $G(T)$ or the infinitely repeated game $G(\infty, \delta)$, a player's strategy specifies the action the player will take in each stage, for each possible history of play through the previous stage.*

Now, we define subgame in this repeated game context.

Definition 3.11. *In the finitely repeated game $G(T)$,* **a subgame** *beginning at stage $t+1$ is the repeated game in which G is played $T-t$ times, denoted $G(T+t)$. There are many subgames that begin at stage $t+1$, one for each of the possible histories of play through stage t.*

In the infinitely repeated game $G(\infty, \delta)$, each subgame beginning at stage $t+1$ is identical to the original game $G(\infty, \delta)$. As in the finite-horizon case, there are as many subgames beginning at stage $t+1$ $G(\infty, \delta)$ as there are possible histories of play through stage t.

To show that the trigger-strategy Nash equilibrium in the infinitely repeated Prisoners' Dilemma is subgame-perfect, we must show that the trigger strategies constitute a Nash equilibrium on every subgame of that infinitely repeated game. Recall that every subgame of an infinitely repeated game is identical to the game as a whole. In the trigger-strategy Nash equilibrium of the infinitely repeated Prisoners' Dilemma, these subgames can be grouped into two classes:

(i) subgames in which all the outcomes of earlier stages have been (R_1, R_2)

(ii) subgames in which the outcome of at least one earlier stage differs from (R_1, R_2).

If the players adopt the trigger strategy for the game as a whole, then

(i) the players' strategies in a subgame in the first class are again the trigger strategy, which we have shown to be a Nash equilibrium of the game as a whole

(ii) the players' strategies in a subgame in the second class are simply to repeat the stage-game equilibrium (L_1, L_2) forever, which is also a Nash equilibrium of the game as a whole.

Thus, the trigger-strategy Nash equilibrium of the infinitely repeated Prisoner's Dilemma is subgame-perfect.

The trigger equilibrium is just one of potentially many subgame perfect equilibria in the repeated Prisoner's Dilemma, for example, playing (L_1, L_2) in every period, regardless of the history of play is also a subgame perfect equilibrium. This equilibrium is not very "cooperative," and it yields a low payoff relative to the trigger profile.

Chapter 4

Static Games of Incomplete Information

4.1 Motivation

When some players do not know the payoffs of the others, the game is said to have *incomplete information*. Many games of interest have incomplete information to at least some extent; the case of perfect knowledge of payoffs is a simplifying assumption that may be a good approximation in some cases.
We now consider some simple examples in which incomplete information matters

Example 4.1. (Entry Game with Incomplete Information) *Consider an industry with two firms: An Incumbent (Player 1) and a potential Entrant (Player 2). Player 1 decides whether to build a new plant, and simultaneously Player 2 decides whether to enter. Imagine that Player 2 is uncertain whether player l's cost of building is* 3 *or* 0*, while player 1 knows her own cost. The payoffs are depicted in Figure 4.1.*

Payoffs if 1's building cost is high

		Player 2: Enter	Player 2: Don't
P 1	Build	$0, -1$	$2, 0$
	Don't Build	$2, 1$	$3, 0$

Payoffs if 1's building cost is low

		Player 2: Enter	Player 2: Don't
P 1	Build	$3, -1$	$5, 0$
	Don't Build	$2, 1$	$3, 0$

Figure 4.1: Entry Game with Incomplete Information

Player 2's payoff depends on whether player 1 builds, but is not directly influenced by player l's cost. Entering is profitable for player 2 if and only if player 1 does not build. Note also that player 1 has a dominant strategy: "Build" if her cost is low and "Don't build" if her cost is high.

Let θ, denote the prior probability Player 2 assigns to Player l's cost being high. Because Player 1 builds if and only if her cost is low, Player 2 enters whenever $\theta > \dfrac{1}{2}$ and stays out if $\theta < \dfrac{1}{2}$. Thus, we can solve the game in Figure 4.1 by the iterated deletion of strictly dominated strategies.

Example 4.2. (Variant of Entry Game) *Let the payoffs be as depicted in Figure 4.2.*

Payoffs if 1's building cost is high:

P 1 \ Player 2	Enter	Don't
Build	$0, -1$	$2, 0$
Don't Build	$2, 1$	$3, 0$

Payoffs if 1's building cost is low:

P 1 \ Player 2	Enter	Don't
Build	$1.5, -1$	$3.5, 0$
Don't Build	$2, 1$	$3, 0$

Figure 4.2: Variant of Game in Figure 4.1

In this new game, "Don't build" is still a dominant strategy for Player 1 when her cost is high. However, when her cost is low, Player l's optimal strategy depends on her prediction of q, the probability that Player 2 enters: Building is better than not building if

$$1.5q + 3.5(1-q) > 2q + 3(1-q) \quad \Rightarrow \quad q < \frac{1}{2}.$$

Thus, Player 1 must try to predict Player 2's behavior to choose her own action, and Player 2 cannot infer Player l's action from his knowledge of Player l's payoffs alone.

Harsanyi (1967–68) proposed that the way to model and understand this situation is to *introduce a prior move by Nature* that determines Player l's "type" (here, her cost). In the transformed game, Player 2's incomplete information about Player l's cost becomes imperfect information about Nature's moves, so the transformed game can be analyzed with standard techniques. The transformation of incomplete information into imperfect information is illustrated in Figure 4.3, which depicts Harsanyi's rendering of the game of Figure 4.2. First, Nature chooses Player l's type. (In the Figure, numbers in brackets are probabilities of Nature's moves.) The Figure incorporates the standard assumption that all players have the same prior beliefs about the probability distribution on Nature's moves.[1] Once this *common-prior* assumption is imposed, we have a standard game, to which Nash equilibrium can be applied. Harsanyi's Bayesian equilibrium (or Bayesian Nash equilibrium) is precisely the Nash equilibrium of the imperfect-information representation of the game.

For instance, in the game of Figure 4.2 (or Figure 4.3), let p denote Player 1's probability of building when her cost is low (Player 1 never builds when her cost is high), and

[1]Although this is a standard assumption, it may be more plausible when Nature's moves represent public events, such as the weather, than when nature's moves model the determination of the players' payoffs and other private characteristics.

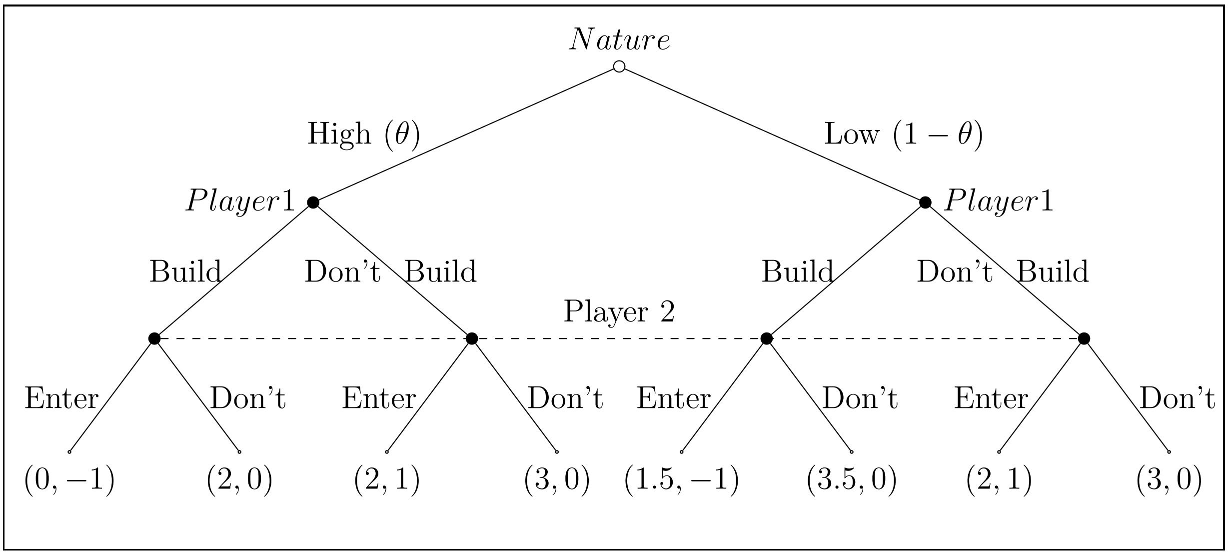


Figure 4.3: Game in Figure 4.2 as a Game with Imperfect Information

let q denote Player 2's probability of entry. The best-response for Player 2 is

$$q = \begin{cases} 1 & \text{if } p < \dfrac{1}{2(1-\theta)}, \\ 0 & \text{if } p > \dfrac{1}{2(1-\theta)}, \\ \{q : 0 \leq q \leq 1\} & \text{if } p = \dfrac{1}{2(1-\theta)}. \end{cases}$$

Similarly, the best-response for the low-cost Player 1 is

$$p = \begin{cases} 1 & \text{if } q < \dfrac{1}{2}, \\ 0 & \text{if } q > \dfrac{1}{2}, \\ \{p : 0 \leq p \leq 1\} & \text{if } q = \dfrac{1}{2}. \end{cases}$$

The search for a *Bayesian equilibrium* boils down to finding a pair (p, q) such that p is optimal for Player 1 with low cost against Player 2 and q is optimal for Player 2 against Player 1 given beliefs θ, and Player 1's strategy. For instance, $(p = 0, q = 1)$ (Player 1 does not build, Player 2 enters) is an equilibrium for any θ, and $(p = 1, q = 0)$ (Player 1 builds if her cost is low, and Player 2 does not enter) is an equilibrium if and only $\theta \leq \frac{1}{2}$.[2]

[2] In this game there is also a mixed strategy equilibrium: $(p = \frac{1}{2(1-\theta)}, q = \frac{1}{2})$.

Example 4.3. *Consider a Cournot duopoly model with inverse demand given by* $p(q) = a - bq$*, where* $q = q_1 + q_2$ *is the aggregate quantity on the market. Firm 1's marginal cost is* c*. Firm 2's marginal cost, however, is* c_H *(high cost) with probability* θ *and* c_L *(low cost) with probability* $1 - \theta$*, where* $c_L < c_H$*. Furthermore, information is asymmetric: Firm 2 knows its cost function and Firm 1's, but Firm 1 knows its cost function and only that Firm 2's marginal cost is* c_H *with probability* θ *and* c_L *with probability* $1 - \theta$*. (Firm 2 could be a new entrant to the industry, or could have just invented a new technology.) All of this is common knowledge: Firm 1 knows that Firm 2 has superior information, Firm 2 knows that Firm 1 knows this, and so on. Both the firms choose their outputs simultaneously.*

Let us look for a pure strategy equilibrium of this game. The problem of Firm 2 when its cost is high is

$$\max_{q_2 \geq 0} q_2[p - c_H] \equiv \max_{q_2 \geq 0} q_2[a - b(q_1 + q_2) - c_H]$$

From this we can get Firm 2's best-response, when its cost is high is

$$q_2^H \equiv BR_2^H(q_1) = \frac{a - c_H}{2b} - \frac{q_1}{2}.$$

Similarly, Firm 2's best-response, when its cost is low is

$$q_2^L \equiv BR_2^L(q_1) = \frac{a - c_L}{2b} - \frac{q_1}{2}.$$

Let us denote Firm 2's output when the cost is high by q_2^H and that when the cost is low by q_2^L.

Now let us consider Firm 1's problem. Firm 1 knows that firm 2's cost is high with probability θ and should anticipate that Firm 2's quantity choice will be q_2^H or q_2^L, depending on Firm 2's cost. Thus, Firm 1's problem is to

$$\max_{q_1 \geq 0} \theta\Big[q_1[a - b(q_1 + q_2^H) - c]\Big] + (1 - \theta)\Big[q_1[a - b(q_1 + q_2^L) - c]\Big].$$

Hence, Firm 1's best-response is

$$\frac{a - c}{2b} - \frac{\theta q_2^H + (1 - \theta) q_2^L}{2}.$$

Solving these we get: $q_1^* = \dfrac{a - 2c}{3b} + \dfrac{\theta c_H + (1 - \theta) c_L}{3b}$

$$q_2^H = \frac{a + c - 2c_H}{3b} + \frac{(1 - \theta)(c_H - c_L)}{6b}, \quad q_2^L = \frac{a + c - 2c_L}{3b} - \frac{\theta(c_H - c_L)}{6b}.$$

4.2 Bayesian Game

Now we formally define an analogue of a strategic game in this environment. Harasanyi was the first to formally define this.

Definition 4.1. *A* **Bayesian game (game of incomplete information)** *is defined by*

- N : *a finite set of players,*
- T_i : *set of* **types** *(signals) for each player* i, *and* $T = \times_{i \in N} T_i$ *is the set of type vectors,*
- p : *a common probability distribution (belief or prior) over* T *with the restriction that* $\pi_i(t_i) := \sum_{t_{-i} \in T_{-i}} p(t_i, t_{-i}) > 0$ *for each* $t_i \in T_i$ *and for each* $i \in N$,
- $A_i(t_i)$: *the set of actions available to each Player* i *with type* t_i,
- $u_i(t_i, a)$: *the payoff assigned by each Player* i *at type profile* $t \equiv (t_1, ..., t_n) \in T$ *when action profile* $a \equiv (a_1, ..., a_n)$, *where each* $a_j \in A_j(t_j)$ *for all* $j \in N$, *is played.*

A Bayesian game proceeds in a sequence where some of the associated uncertainties are resolved.

- The type vector $t \in T$ is chosen (by nature) using the probability distribution p.
- Each player $i \in N$ observes his own type t_i but does not know the types of other agents.
- After observing their types, each player i plays an action $a_i \in A_i(t_i)$.
- Each player i receives an utility equal to $u_i(t, a)$ when the type profile realized is $t \equiv (t_1, ..., t_n)$ and the action profile is $a \equiv (a_1, ..., a_n)$.

Let us try to understand the notion of type more clearly.

4.2.1 The Notion of Type

In the examples of Section 4.1, a player's "type" – his private information was simply his cost. More generally, the "type" of a player embodies *any private information* (more precisely, any information that is not common knowledge to all players) that is relevant to the player's decision making. This may include, in addition to the player's payoff function, his beliefs about other players' payoff functions, his beliefs about what other players believe his beliefs are, and so on.

We have already seen examples where the players' types are identified with their payoff functions. In the following example the type includes more than this.

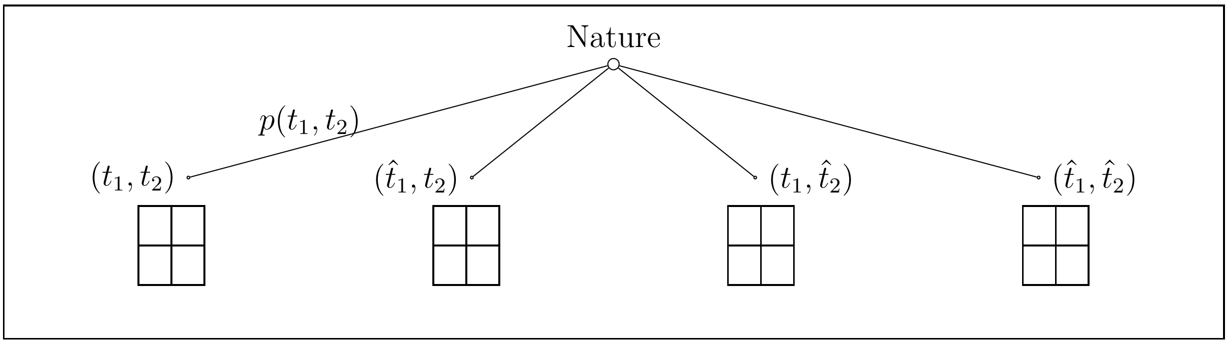


Figure 4.4: A Bayesian Game

Example 4.4. *Consider disarmament talks between two negotiators. Player 2's objective function is public information; Player 1 is uncertain whether Player 2 knows Player l's objectives. To model this, suppose that Player 1 has two possible types – a "tough" type, who prefers no agreement to making substantial concessions, and a "weak" type, who prefers any agreement to none at all – and that the probability that Player 1 is tough is* π_1*. Furthermore, suppose that Player 2 has two types – "informed" who observes Player l's type, and "uninformed," who does not observe Player l's type. The probability that Player 2 is informed is* π_2*, and Player 1 does not observe player 2's type.*

Example 4.5. *Figure 4.4 illustrates a Bayesian game for two players with the type set of Player 1 being* $\{t_1, \hat{t}_1\}$ *and that of Player 2 being* $\{t_2, \hat{t}_2\}$*.*

As the Figure shows, a Bayesian game can be described by a sequence of moves, where the first move is by Nature determining the type vector of players. Figure 4.4 shows the four possible type vectors. Once the type vectors are realized Players know the actions available to them (but not to others as they do not know the types of others). Hence, there is still uncertainty about the game being played.

In most of the examples, we will make the assumption that for all t_{-i}, $t^{'}_{-i}$

$$u_i((t_i, t_{-i}), a) = u_i((t_i, t^{'}_{-i}), a) \text{ for all } a, \text{ for all } t_i, \text{ for all } i \in N.$$

This is called a **private values** model. It rules out the possibility that a player's utility depends directly on the type of other players. Notice that the action chosen by a player may depend on his type in the game, and hence, indirectly, Player i's utility will depend on the type of other players (though the actions chosen by other players).

4.2.2 The Notion of Strategy and Utility

Recall that a player's strategy is a complete plan of action, specifying a feasible action in every contingency in which the player might be called on to act. Given the timing of a static Bayesian game, in which Nature begins the game by drawing the players' types, a (pure) strategy for Player i must specify a feasible action for each of player i's possible types.

Definition 4.2. *In the static Bayesian game, a strategy for Player i is a function $s_i(t_i)$, where for each type $t_i \in T_i$, $s_i(t_i)$ specifies the action from the feasible set A_i that type t_i would choose if drawn by Nature.*

In a static Bayesian game the strategy spaces are constructed from the type and action spaces: Player i's set of possible (pure) strategies, S_i is the set of all possible functions with domain T_i, and range A_i. In a **separating strategy**, for example, each type $t_i \in T_i$, chooses a different action a_i from A_i. In a **pooling strategy**, in contrast, all types choose the same action.

Now, it may seem unnecessary to require Player i's strategy to specify a feasible action for each of Player i's possible types. After all, once Nature has drawn a particular type and revealed it to a player, it may seem that the player need not be concerned with the actions he or she would have taken had nature drawn some other type. On the other hand, Player i needs to consider what the other players will do, and what they will do depends on what they think Player i will do, for each $t_i \in T_i$. Thus, in deciding what to do once one type has been drawn, Player i will have to think about what he or she would have done if each of the other types in T_i had been drawn.

Consider the asymmetric-information Cournot game in example 4.3. We argued that the solution to the game consists of three quantity choices: q_1^*, q_2^H and q_2^L. In terms of the definition of a strategy just given, the pair (q_2^H, q_2^L) is Firm 2's strategy and q_1^* is Firm I's strategy. It is easy to imagine that Firm 2 will choose different quantities depending on its cost. It is equally important to note, however, that firm 1's single quantity choice should take into account that Firm 2's quantity will depend on Firm 2's cost in this way. Thus, if our equilibrium concept is to require that Firm I's strategy be a best response to Firm 2's strategy, then Firm 2's strategy must be a pair of quantities, one for each possible cost type, else Firm 1 simply cannot compute whether its strategy is indeed a best response to Firm 2's.

More generally, we would not be able to apply the notion of Nash equilibrium to Bayesian games if we allowed a player's strategy not to specify what the player would do if some types were drawn by nature. This argument is analogous to one from Chapter 3: It may have seemed unnecessary to require Player i's strategy in a dynamic game of complete information to specify a feasible action for each contingency in which Player i might be called on to move, but we could not have applied the notion of Nash equilibrium to dynamic games of complete information if we had allowed a player's strategy to leave the player's actions in some contingencies unspecified.

What is the payoff of Player i from a strategy profile $s \equiv (s_1, ..., s_n)$? There are two ways to think about it: *ex-ante payoff*, which is computed before realization of the type, and *interim payoff*, which is computed after realization of the type. Ex-ante payoff from strategy profile s is

$$EU_i(s) := \sum_{t \in T} p(t) u_i(t, (s_1(t_1), ...s_n(t_n))).$$

Here, if type profile t is realized, then action profile $(s_1(t_1), ...s_n(t_n))$ is played according to the strategy profile s. Hence, the payoff realized by Player i at type profile t is just $u_i(t, (s_1(t_1), ...s_n(t_n)))$. Then, $EU_i(s)$ computed using expectation from this.

The interim payoffs are computed by *updating beliefs* after realizing the types. In particular, once Player i knows his type to be $t_i \in T_i$, he computes his conditional probabilities as follows. For every $t_{-i} \in T_{-i}$,

$$p_i(t_{-i}|t_i) := \frac{p(t_{-i}, t_i)}{\sum_{t'_{-i} \in T_{-i}} p(t'_{-i}, t_i)} = \frac{p(t_{-i}, t_i)}{\pi_i(t_i)},$$

where we will denote $\pi_i(t_i) \equiv \sum\limits_{t'_{-i} \in T'_{-i}} p(t'_{-i}, t_i)$ and note that it is positive by our assumption. The interim payoff of Player i with type t_i from a strategy profile s_{-i} of other players and when he takes action $a_i \in A_i(t_i)$ is thus

$$EU_i(a_i, s_{-i})|t_i) := \sum_{t'_{-i} \in T_{-i}} p_i(t'_{-i}|t_i) u_i(t, (a_i, s_{-i}(t'_{-i}))).$$

If the beliefs are *independent*, then observing own type gives no extra information to the players. Hence, no updating of prior belief is required by the players.

An easy consequence of this definition is the following. Consider Player i and a strategy profile (s_i, s_i)

$$\begin{aligned}
\sum_{t_i \in T_i} EU_i((s_i(t_i), s_{-i})|t_i)\pi_i(t_i) &= \sum_{t_i \in T_i} \pi_i(t_i) \sum_{t_{-i} \in T_{-i}} p_i(t_{-i}|t_i) u_i(t, (s_i(t_i), s_{-i}(t_{-i}))) \\
&= \sum_{t \in T} p(t) u_i(t, (s_i(t_i), s_{-i}(t_{-i}))) = EU_i(s).
\end{aligned}$$

Note: The above expressions are for finite type spaces, but similar expressions (using integrals) can also be written with infinite type spaces.

4.3 Bayesian Equilibrium

As we saw, there are two points at which a player may evaluate his utility: ex-ante or interim. Depending on that the notion of equilibrium can be defined. The ex-ante notion coincides with the idea of a Nash equilibrium.

Definition 4.3. *A strategy profile s^* is a* **Nash equilibrium** *in a Bayesian game if for each player i and each pure strategy s_i,*

$$EU_i(s^*_i, s^*_{-i}) \geq EU_i(s_i, s^*_{-i}).$$

There is also an interim way of defining the equilibrium. This is called the *Bayesian equilibrium*, and is the common way of defining equilibrium in Bayesian games.

Definition 4.4. *A strategy profile s^* is a* **Bayesian equilibrium** *in a Bayesian game if for each player i, each type $t_i \in T_i$, and each action $a_i \in A_i(t_i)$,*

$$EU_i((s_i^*(t_i), s_{-i}^*)|t_i) \geq EU_i((a_i, s_{-i}^*)|t_i) \ \ \forall t_i \in T_i.$$

Informally, it says that a player i of type t_i maximizes his expected/interim payoff by following s_i^* given that all other players follow s_{-i}^*.

When the type space is finite, it can be shown that a strategy profile is a Nash equilibrium if and only if it is a Bayesian equilibrium. In other words, a player has a profitable deviation in Bayesian game before he learns his type if and only if he has a profitable deviation after he learns his type. The equivalence result needs type spaces to be finite. In general, games where type space is not finite, a Bayesian equilibrium will continue to imply a Nash equilibrium but the converse need not hold. So, we will use the solution concept Bayesian equilibrium in all the Bayesian games that we analyze.

But do all Bayesian games admit a Bayesian equilibrium? Which Bayesian games admit a Bayesian equilibrium? There is a long literature on this topic, which we will skip. Just like Nash equilibrium, there is a well-behaved class of games that admit a Bayesian equilibrium: A *finite* static Bayesian game (i.e., a game in which n is finite and $(A_1, ..., A_n)$ and $(T_1, ..., T_n)$ are all finite sets) there exists a Bayesian Nash equilibrium, perhaps in mixed strategies.

4.4 Mixed Strategies Revisited

In chapter 2 we saw that simultaneous-move games of complete information often admit mixed-strategy equilibria. One may argue that "real-world decision makers do not flip coins." However, Harsanyi (1973) suggested that Player j's mixed strategy represents Player j's uncertainty about j's choice of a pure strategy, and that j's choice in turn depends on the realization of a small amount of private information. We can now give a more precise statement of this idea: A mixed strategy Nash equilibrium in a game of complete information can (almost always) be interpreted as a pure-strategy Bayesian Nash equilibrium in a closely related game with a little bit of incomplete information.

Recall that in the Battle of the Sexes there are two pure-strategy Nash equilibria (F, F) and (M, M) and a mixed strategy Nash equilibrium in which Player 1 (Woman) plays F with probability $\dfrac{2}{3}$ and Player 2 (Man) plays M with probability $\dfrac{2}{3}$.

Now consider the following example where the payers are not quite sure of each other's payoffs.

Example 4.6. (Battle of Sexes with Incomplete Information) *Suppose Player 1's payoff if both go to the football match be $2 + t_1$, where t_1 is private information of Player 1; Player 2's payoff if both go to the movie be t_2, where t_2 is privately known by Player*

2; and t_1 *and* t_2 *are independent draws from a uniform* $[0, x]$. *All the other payoffs are the same.*

In formal terms the action space is $A_1 = A_2 = \{F, M\}$, the type spaces are $T_1 = T_2 = [0, x]$, the beliefs are $p_1(t_2) = p_2(t_1) = \dfrac{1}{x}$ for all t_1 and t_2, and the payoffs are as follows

		Player 2: F	Player 2: M
Player 1	F	$2 + t_1, 1$	$0, 0$
	M	$0, 0$	$1, 2 + t_2$

Figure 4.5: The Battle of Sexes with Incomplete Information

We will construct a pure-strategy Bayesian Nash equilibrium of this incomplete-information version of the Battle of the Sexes in which Player 1 plays F if t_1 exceeds a critical value, w, and plays M otherwise and Player 2 plays M if t_2 exceeds a critical value, v, and plays F otherwise. In such an equilibrium, Player 1 plays F with probability $\dfrac{x - w}{x}$ and Player 2 plays M with probability $\dfrac{x - v}{x}$. We will show that as the incomplete information disappears (i.e., as x approaches zero), the players' behavior in this pure-strategy Bayesian Nash equilibrium approaches their behavior in the mixed-strategy Nash equilibrium in the original game of complete information. That is, both $\dfrac{x - w}{x}$ and $\dfrac{x - v}{x}$ approach $\dfrac{2}{3}$ as x approaches zero.

Suppose Player 1 and Player 2 play the strategies just described. For a given value of x, we will determine values of w and v such that these strategies are a Bayesian Nash equilibrium. Given Player 2's strategy, Player 1's expected payoffs from playing F and from playing M are

$$\frac{v}{x} \cdot (2 + t_1) \qquad \text{and} \quad \frac{x - v}{x} \cdot 1,$$

respectively. Thus playing F is optimal if and only if

$$\frac{v}{x} \cdot (2 + t_1) \geq \frac{x - v}{x} \cdot 1 \quad \Rightarrow \; t_1 \geq \frac{x}{v} - 3 = w.$$

Similarly, given Player 1's strategy, Player 2's expected payoffs from playing F and from playing M are

$$\frac{x - w}{x} \cdot 1 \qquad \text{and} \quad \frac{w}{x} \cdot (2 + t_2)$$

respectively. Thus playing M is optimal if and only if

$$\frac{w}{x} \cdot (2 + t_2) \geq \frac{x - w}{x} \cdot 1 \quad \Rightarrow \; t_2 \geq \frac{x}{w} - 3 = v.$$

Solving them simultaneously we get $w = v$ and $w^2 + 3w - x = 0$. Solving the quadratic then shows that the probability that Player 1 plays F, namely $\dfrac{x - w}{x}$, and the probability that Player 2 plays M, namely $\dfrac{x-v}{x}$, both equal

$$1 - \frac{-3+\sqrt{9+4x}}{2x}$$

which approaches $\dfrac{2}{3}$ as x approaches zero. Thus, as the incomplete information disappears, the players' behavior in this pure strategy Bayesian Nash equilibrium of the incomplete-information game approaches their behavior in the mixed-strategy Nash equilibrium in the original game of complete information.

4.5 The Market for "Lemons"

This story is based on a paper that was published in 1970 by the American economist George Akerloff (1940–). (George Akerloff, Michael Spence (1943–), and Joseph Stiglitz (1943–) won the 2001 Nobel Prize in Economics for their work on markets with asymmetric information.)

Consider the market for used cars. Considerable uncertainty is attached to any used car. It may break down and require a new engine or transmission tomorrow, or it may run perfectly, needing only occasional routine maintenance, for the next ten years. When people are buying and selling used cars, the sellers usually have better information about the cars' reliability than the buyers. The sellers know much more about the probabilities of nasty and costly mechanical failures that may happen next month or next year.

Example 4.7. *Suppose there are two kinds of used cars: good quality cars (peaches) and bad quality cars (lemons) and the proportion of peaches and lemons be θ and $1-\theta$ respectively. The owners perfectly know the quality or type of their cars. Buyers and sellers, both, are risk neutral.*

Given her knowledge of the expected costs of future repairs, the owner of each type of car has a *reservation price* for her car. Let r^L and r^P be reservation prices for a lemon and a peach respectively, where $r^L < r^P$. So, if a buyer sell his type i car at price p^i, then his utility will be $p^i - r^i$.

Buyers also have *willingness to pay* for either type of car, contingent on the type: Let that be v^L for a lemon, and v^P for a peach. To make things interesting, let us assume $r^L < v^L$ and $r^P < v^P$.

Benchmark Case: Potential buyers know the type – they can distinguish between peaches and lemons. So, if a buyer buys type i car at price p^i, then his utility would be $v^i - p^i$.

The market price for lemons p^L will be in between r^L and v^L and a market price for peaches p^P will be in between r^P and v^P. The *markets would clear*, and the result would be *efficient or Pareto optimal.*

Asymmetric Information: Potential buyers cannot distinguish between peaches and lemons, they only know the proportion of peaches to lemons, that is θ. This is common knowledge.

Can there be different prices for peaches and lemons? No, because if $p^i > p^j$, where $i, j \in \{L, M\}$, then owner of j type car will report his car to be of type i. Hence, when there is asymmetry in information there can be only one price, let that be p.

So, at p a buyer will expect to buy a peach with probability θ and a lemon with probability $1-\theta$. Hence, the buyer would be willing to pay $v \equiv \theta v^P + (1-\theta)v^L$.
An owner of peach will be willing to sell at v if and only if

$$v \geq r^P \quad \Rightarrow \; \theta v^P + (1-\theta)v^L \geq r^P \quad \Rightarrow \; \theta \geq \frac{r^P - v^L}{v^P - v^L}.$$

This gives us a stark result – even though buyers are willing to pay for a peach more than its reservation price, when $\theta < \dfrac{r^P - v^L}{v^P - v^L}$, none of the peach will be sold.

Exercise 4.1. *Consider a used car market in which a fraction q of the cars $(0 \leq q \leq 1)$ are in good condition and $1-q$ are in bad condition (lemons). The seller (Player 2) knows the quality of the car he is offering to sell while the buyer (Player 1) does not know the quality of the car that he is being offered to buy. Each used car is offered for sale at the price of \$$p$ (in units of thousands of dollars). The payoffs to the seller and the buyer, depending on whether or not the transaction is completed, are described in the following tables:*

Car in Good Condition

		Player 2 *Sell*	*Player 2* *Don't*
P 1	*Buy*	$6-p, p$	$0, 5$
	Don't	$0, 5$	$0, 5$

Car in Bad Condition

		Player 2 *Sell*	*Player 2* *Don't*
P 1	*Buy*	$4-p, p$	$0, 0$
	Don't	$0, 0$	$0, 0$

Depict this situation as a Bayesian game with incomplete information, and for each pair of parameters p and q, find all the Bayesian equilibria.

4.6 Auction

We now study different forms of auction. There are a number of ways in which bidders may value an object up for auction. The main distinction in such auction environments is based on the difference between *common* and *private*-value objects. In a *common-value*, or *objective-value*, auction, the value of the object is the *same* for all the bidders, but each bidder generally knows only an imprecise estimate of it. Bidders may have some sense of the distribution of possible values, but each must form her own estimate before bidding. For example, an oil-drilling tract has a given amount of oil that should produce the same revenue for all companies, but each company has only its own expert's estimate of the amount of oil contained under the tract.

In a common-value auction, each bidder should be aware of the fact that other bidders possess some (however sketchy) information about an object's value, and he should attempt to infer the contents of that information from the actions of rival bidders. In addition, he should be aware of how his own actions might signal his private information to those rival bidders. When bidders' estimates of an object's value are influenced by their beliefs about other bidders estimates, we have an environment in which bids are said to be correlated with each other.

In a *private-value*, or *subjective-value*, auction, bidders each determine their own individual value for an object. In this case, bidders place *different* values on an object. Bidders know their own private valuations in such auction environments but do not know each other's valuations of an object. Similarly, the seller does not know any of the bidders' valuations.

Next we start with *private* value auction. We find symmetric Bayesian Nash equilibrium in first-price auction. Then, we compare revenue generated in the first and second price auctions. After that we address double auction (or bilateral trading). We, then, turn our attention to *common value* auction and introduce an interesting concept called *winner's curse.*

4.6.1 Private Value Auction

4.6.1.1 First Price Auction

Example 4.8. *Consider the following first-price, sealed-bid auction. There are two bidders, labeled $i = 1, 2$. Bidder i has a valuation v_i for the good – that is, if bidder i gets the good and pays the price p, then ,i's payoff is $v_i - p$. The two bidders' valuations are independently and uniformly distributed on $[0, 1]$. Bids are constrained to be nonnegarive. The bidders simultaneously submit their bids. The higher bidder wins the good and pays the price she bid; the other bidder gets and pays nothing. In case of a tie, the winner is determined by a flip of a coin. The bidders are risk-neutral. All of this is common knowledge.*

In order to formulate this problem as a Bayesian game, we must identify the action spaces, the type spaces, the beliefs, and the payoff functions. Player i's action is to submit a (nonnegative) bid, b_i (that is action space is $A_i = [0, \infty)$), and her type is her valuation, v_i (type space is $T_i = [0, 1]$). Because the valuations are independent, Player i believes that v_j is uniformly distributed on $[0, 1]$, no matter what the value of v_i. Finally, Player i's payoff function is

$$u_i(b_1, b_2; v_1, v_2) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \dfrac{v_i - b_i}{2} & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j. \end{cases}$$

Unlike the second-price (Vickrey) auction, the first-price auction has no weakly dominant strategy (verify). To see this, note that when bidder i bids her valuation (v_i), she gets a payoff of zero when she wins the object. Hence, her expected payoff from bidding her valuation is zero. If she bids a little less than v_i, she runs the risk of losing the object should a rival bidder make a bid above her but below v_i. But as long as she does not bid so low that this outcome is guaranteed, she has a positive probability of making a positive profit. Her optimal bidding strategy entails *shading* her bid. We will see this formally, but the intuition is simple. An increase in shading (a lowering of her bid from v_i) provides both an advantage and a disadvantage; it increases her profit margin if she wins the object, but it also lowers her chances of being the high bidder and therefore of actually obtaining the object. Her bid is optimal when the last bit of shading just balances these two effects.

We derive Bayesian (Nash) equilibrium. Recall that in a Bayesian game, a strategy is a function from types to actions. Thus, a strategy for Player i is a function $b_i(v_i)$ specifying the bid that each of i's types (i.e., valuations) would choose. In a Bayesian Nash equilibrium, Player 1's strategy $b_1(v_1)$ is a best response to Player 2's strategy $b_2(v_2)$, and vice versa. Formally, the pair of strategies $(b_1(v_1), b_2(v_2))$ is a Bayesian equilibrium if for each $v_i \in [0, 1]$, $b_i(v_i)$ solves

$$\max_{b_i} [v_i - b_i] Prob(b_i > b_j(v_j)) + \frac{1}{2}[v_i - b_i] Prob(b_i = b_j(v_j)).$$

In particular, we find out *symmetric* Bayesian (Nash) equilibrium, under the assumption that the players' strategies are *strictly increasing and differentiable.* In other words, we want to find out a bidding function $b(\cdot)$ which is strictly increasing and differentiable such that if Bidder j follows $b(v_j)$ strategy then the expected payoff maximizing strategy (over all strategies, including non-symmetric ones) for Bidder i must be $b(v_i)$ when his value is v_i. Note that if Bidder i with value v_i bids $b(v_i)$, and since Bidder j is using $b(\cdot)$ strategy, *increasing $b(\cdot)$ ensures that the probability of winning for Bidder i is equal to the probability that v_i is the highest value.*

We now define the notion of a symmetric (Bayesian) equilibrium in this case as follows. Suppose a bidder i bids b_i and other bidders follow a symmetric strategy $b(\cdot)$. Then this bidder wins if Bidder j bids less than b_i which is possible if her value is $b^{-1}(b_i)$. So, the probability of a bidder winning the object by bidding b_i when Player j follows $b(\cdot)$ strategy is $\dfrac{b^{-1}(b_i) - 0}{1 - 0} = b^{-1}(b_i)$. If $b_i = b(v_i)$, i.e., Bidder i also bids according to $b(\cdot)$ strategy, then this probability is just v_i. Hence, the notion of Bayes-Nash equilibrium reduces to the following definition.

Definition 4.5. *A strategy profile* $b : [0,1] \to \mathbb{R}^+$ *for both the agents is a symmetric Bayesian equilibrium if for every bidder* i *and every type* $v_i \in [0,1]$

$$Prob(b^{-1}(b(v_i)) > v_j)[v_i - b(v_i)] \geq Prob(b^{-1}(b_i) > v_j)[v_i - b_i] \quad \Rightarrow v_i[v_i - b(v_i)] \geq b^{-1}(b_i)[v_i - b_i]$$

$$\forall b_i \in \mathbb{R}^+.$$

We now show that the unique symmetric Bayesian Nash equilibrium is each player submits a bid equal to half her valuation, that is $b(v_i) = \dfrac{v_i}{2}$.

Suppose player j adopts the strategy $b(\cdot)$, then Player i's optimal bid solves

$$\max_{b_i \geq 0} \, [v_i - b_i] Prob(b_i > b(v_j)).$$

The best-response of Player i to the strategy $b(\cdot)$ played by bidder j is

$$-b^{-1}(b_i) + (v_i - b_i)\frac{d}{db_i} b^{-1}(b_i) = 0.$$

If the strategy $b(\cdot)$ is to be a symmetric Bayesian Nash equilibrium, we require that the solution to the first order condition be $b(v_i)$: that is, for each of Bidder i's possible valuations, Bidder i does not wish to deviate from the strategy $b(\cdot)$, given that bidder j plays this strategy. To impose this requirement, we substitute $b_i = b(v_i)$ into the first-order condition, yielding:

$$\begin{aligned} &- v_i + (v_i - b(v_i))\frac{1}{b'(v_i)} = 0 \\ &\Rightarrow b'(v_i)v_i + b(v_i) = v_i. \end{aligned}$$

Integrating both sides of the equation we get

$$b(v_i)v_i = \frac{1}{2}v_i^2 + k,$$

where k is a constant of integration. To eliminate k, we need a boundary condition. Fortunately, simple economic reasoning provides one: No player should bid more than his or her valuation. Thus, we require $b(v_i) \leq v_i$ for every v_i. In particular, we require $b(0) \leq 0$. Since bids are constrained to be nonnegative, this implies that $b(0) = 0$, so $k = 0$ and $b(v_i) = \dfrac{v_i}{2}$ as claimed.

Observe, the technique we use can easily be extended to a broad class of valuation distributions, as well as the case of n bidders.

4.6.1.2 Revenue Equivalence

Now we can compare the selling prices – the revenues accruing to the seller – in the first and second price auctions.
In a first-price auction, the winner pays what he bids and thus the *ex ante* expected payment by a bidder with value (v_i) is

$$\pi_i^I(v_i) = Prob(b(v_i) > b(v_j)) \cdot b(v_i) = v_i \cdot \frac{v_i}{2} = \frac{v_i^2}{2}.$$

Now, consider the second-price auction. A bidder with value v_i pays b_j if he is the highest valued bidder. Hence, his *ex ante* expected payment is the probability that he has the highest value (which is v_i) times the conditional expected value of the highest of other values:

$$\pi_i^{II}(v_i) = Prob(b(v_i) > b(v_j)) \cdot E(v_j | v_j < v_i) = \frac{v_i^2}{2}.$$

The expected revenue accruing to the seller R^a (where $a = I$ or II) is just 2 times the *ex ante* expected payment of an individual bidder, so,

$$R^a = 2 \cdot \int_0^1 \frac{v_i^2}{2} dv_i = 2\Big[\frac{v_i^3}{6}\Big]_0^1 = \frac{1}{3}.$$

More generally, the following result is known as *revenue equivalence theorem.*

Theorem 4.1. *Suppose values of bidders are distributed independently and identically. Then the expected revenue from the first-price auction and the second- price auction is the same.*

4.6.1.3 Double Auction/ Bilateral Trading

The double auction/ bilateral trading is one of the simplest model to study Bayesian games. It involves two players: A buyer (b) and a seller (s). The seller can produce a good with cost c and the buyer has a value v for the good. Suppose both the value and the cost are distributed uniformly in $[0, 1]$.

Now, consider the following Bayesian game. The buyer announces a price p_b that he is willing to pay and the seller announces a price p_s that she is willing to accept. Trade occurs if $p_b > p_s$ at a price equal to $\frac{p_b + p_s}{2}$. If $p_b < p_s$, then no trade occurs.

- *Type:* The type of the buyer is his value $v \in [0, 1]$ and the type of the seller is his cost $c \in [0, 1]$.
- *Strategy:* A strategy for each agent is to announce a price given their types. In other words, the strategy of the buyer is a map $p_b : [0, 1] \to \mathbb{R}$ and the strategy of the seller is a map $p_s : [0, 1] \to \mathbb{R}$.
- *Payoffs:* If no trade occurs, then both the agents get zero payoff. If trade occurs at price p, then the buyer gets a payoff of $v - p$ and the seller gets a payoff of $p - c$.

A pair of strategies $(p_b(v), p_s(c))$ is a *Bayesian Nash equilibrium* if the following two conditions hold.

- For each $v \in [0,1]$, $p_b(v)$ solves

$$\max_{p_b} \Big[v - \frac{p_b + E(p_s(c)|p_b \geq p_s(c))}{2}\Big] Prob(p_b \geq p_s(c)), \tag{4.6.1}$$

  where $E(p_s(c)|p_b \geq p_s(c))$ is the expected price the seller will demand, conditional on the demand being less than the buyer's offer of p_b.

- For each $c \in [0,1]$, $p_s(c)$ solves

$$\max_{p_s} \Big[\frac{p_b + E(p_b(v)|p_b(v) \geq p_s)}{2} - c\Big] Prob(p_b(v) \geq p_s), \tag{4.6.2}$$

  where $E(p_b(v)|p_b(v) \geq p_s)$ is the expected price the the buyer will offer, conditional on the offer being greater than the seller's demand of p_s.

We now derive a *linear Bayesian Nash equilibrium* of the double auction. We do not restrict the players' strategy spaces to include only linear strategies. Rather, we allow the players to choose arbitrary strategies but ask whether there is an equilibrium that is linear. Many other equilibria exist besides the one-price equilibria and the linear equilibrium, but the linear equilibrium has interesting efficiency properties, which we describe later.

Suppose the seller's strategy is $p_s(c) = \alpha_s + \beta_s c$. Then p_s is uniformly distributed on $[\alpha_s, \alpha_s + \beta_s]$, so (4.6.1) becomes

$$\max_{p_b} \Big[v - \frac{1}{2}\Big[p_b + \frac{\alpha_s + p_b}{2}\Big]\Big] \frac{p_b - \alpha_s}{\beta_s},$$

the first order condition for which yields

$$p_b = \frac{2}{3}v + \frac{1}{3}\alpha_s. \tag{4.6.3}$$

Thus, if the seller plays a linear strategy, then the buyer's best response is also linear. Analogously, suppose the buyer's strategy is $p_b(v) = \alpha_b + \beta_b v$. Then p_b is uniformly distributed on $[\alpha_b, \alpha_b + \beta_b]$, so (4.6.2) becomes

$$\max_{p_s} \Big[\frac{1}{2}\Big[p_s + \frac{p_s + \alpha_b + \beta_b}{2}\Big] - c\Big] \frac{\alpha_b + \beta_b - p_s}{\beta_b},$$

the first order condition for which yields

$$p_s = \frac{2}{3}c + \frac{1}{3}(\alpha_b + \beta_b). \tag{4.6.4}$$

Thus, if the buyer plays a linear strategy, then the seller's best response is also linear. If the players' linear strategies are to be best responses to each other, (4.6.3) implies that $\beta_b = \frac{2}{3}$ and $\alpha_b = \frac{\alpha_s}{3}$ and (4.6.4) implies $\beta_s = \frac{2}{3}$, and $\alpha_s = \frac{\alpha_b + \beta_b}{3}$. Therefore, the linear equilibrium strategies are

$$p_b^*(v) = \frac{2}{3}v + \frac{1}{12} \quad \text{and} \quad p_s^*(c) = \frac{2}{3}c + \frac{1}{4}.$$

One notable feature of this equilibrium is that trade occurs when $p_b^*(v) > p_s^*(c)$, which is equivalent to requiring $\frac{2}{3}v + \frac{1}{12} > \frac{2}{3}c + \frac{1}{4}$. This gives $v - c > \frac{1}{4}$. Note that efficiency will require trade to happen when $v > c$ – such trades will be possible if there is complete information. Hence, there is some loss in efficiency due to incomplete information. This is in general an impossibility – you cannot construct any Bayesian game whose equilibrium will have efficiency in Bayesian equilibrium in this model. The region of trade in this particular equilibrium and efficiency loss is shown in Figure 4.6.

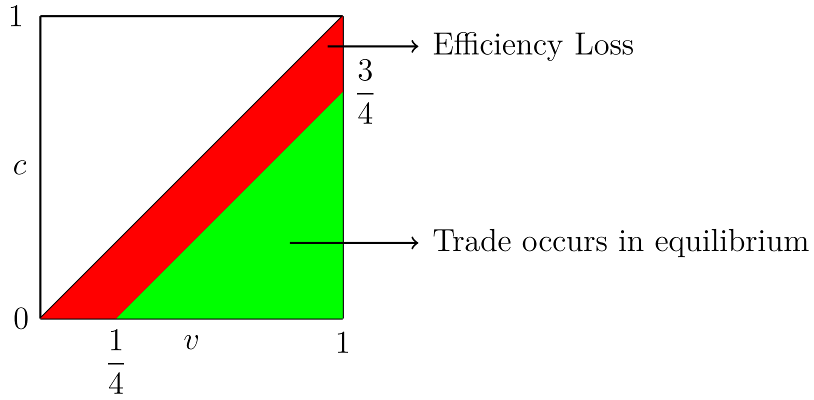


Figure 4.6: Efficiency Loss in Double Auction with Incomplete Information

4.6.2 Common Value Auction and Winner's Curse

Example 4.9. *There are two bidders (players 1 and 2), and they have the same valuation of the object being auctioned. Let Y denote this valuation. Suppose that $Y = y_1 + y_2$, where y_1 and y_2 are uniformly distributed between 0 and 10. Player 1 privately observes the signal y_1 and Player 2 privately observes the signal y_2. The players then engage in a first-price, sealed-bid auction to determine who gets the object.*

As a first step in the analysis, consider the strategy in which Player i bids $b_i = y_i + 5$. That is, Player i's bid is his expected valuation of the object, given his own signal y_i.

(Note that the expected value of y_j is 5.) One might think that this strategy should yield an expected payoff of zero conditional on winning the auction, but this is not so. There is an important subtlety: Conditional on winning the auction, Player i learns something about Player j's signal y_j.

To see how this works, suppose both players use the strategy $b_1 = y_i + 5$, and Player 1 happens to get the signal $y_1 = 8$. Player 1 will then bid $b_1 = 13$, thinking that y_2 is 5 on average. Will player 1 break even? No. Player 1 wins the object only in the event that Player 2's bid is less than 13. Considering that $b_2 = y_2 + 5$, Player 2 bids less than 13 only if $y_2 < 8$. Conditional on $y_2 < 8$, its expected value is 4. Thus, conditional on winning the auction, Player 1 must lower his estimate of y_2 and, along with it, the valuation Y.

The logic just described is known as *the winner's curse.* A player wins when the other players bid less, but this implies that the other bidders must have received relatively bad signals of the object's valuation. The strategic implication is that one should factor in that winning yields information, and this information should be used in formulating the expected valuation.

The Bayesian Nash equilibrium of this common-value auction is the strategy profile in which $b_1 = y_1$ and $b_2 = y_2$, so the players are quite cautious. To confirm this equilibrium, let us calculate Player l's optimal bid as a function of his signal, assuming that Player 2's bidding rule is $b_2 = y_2$. (The same analysis applies to Player 2.) Note that for any signal y_1 and bid b_1, Player l's expected payoff is the probability of winning times the expected payoff conditional on winning. The probability of winning is $\frac{b_1}{10}$ because this is the probability that Player 2 bids below b_1 (owing to $b_2 = y_2$ and that y_2 is uniformly distributed on $[0, 10]$). The expected payoff conditional on winning is

$$y_1 + \frac{b_1}{2} - b_1,$$

because $\frac{b_1}{2}$ is the expected value of y_2 conditional on $y_2 < b_1$. Multiplying these, we see that Player l's expected payoff from bidding b_1 is

$$\frac{b_1}{10}\Big(y_1 + \frac{b_1}{2} - b_1\Big) = \frac{b_1}{10}(y_1 - \frac{b_1}{2}),$$

which is maximized by setting $b_1 = y_1$.

The new wrinkle introduced in the common-value environment is that a player must consider the informational content of winning the auction, as it pertains to the signals that other players received.

Chapter 5

Dynamic Games of Incomplete Information[1]

The concept of subgame perfection, introduced in chapter 3, has no bite in games of incomplete information, even if the players observe one another's actions at the end of each period: Since the players do not know the others' types, the start of a period *does not* form a well-defined subgame until the players' posterior beliefs are specified, and so we cannot test whether the continuation strategies are a Nash equilibrium.

In this chapter we introduce another equilibrium concept— Perfect Bayesian equilibrium. As we consider progressively richer games, we progressively strengthen the equilibrium concept, in order to rule out implausible equilibria in the richer games that would survive if we applied equilibrium concepts suitable for simpler games. In each case, the stronger equilibrium concept differs from the weaker concept only for the richer games, not for the simpler games. In particular, Perfect Bayesian equilibrium is equivalent to Bayesian Nash equilibrium in static games of incomplete information, equivalent to Subgame-Perfect Nash equilibrium in dynamic games of complete and perfect information, and equivalent to Nash equilibrium in static games of complete information.

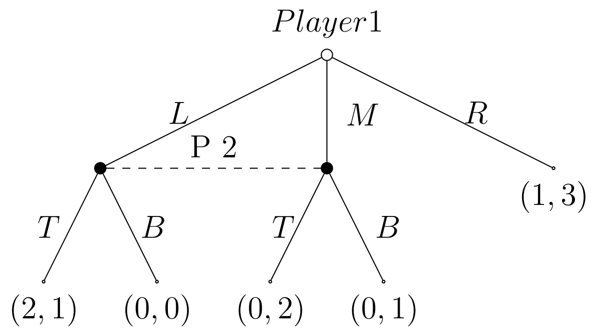
Perfect Bayian equilibrium was invented in order to refine (i.e. strengthen the requirements of) Bayesian Nash equilibrium. Just as we imposed sugame perfection in dynamic games of complete information because Nash equilibrium failed to capture the idea that threats and promises should be credible, we now restrict attention to Perfect Bayesian equilibrium in dynamic games of incomplete information because Bayesian Nash equilibrium suffers from the same flaw. Recall that if the players' strategies are to be a Subgame-Perfect Nash equilibrium, they must not only be a Nash equilibrium for the entire game but must also constitute a Nash equilibrium in every subgame. In this chapter we replace the idea of subgame with the more general idea of continuation game – the latter can begin at any complete information set (whether singleton or not), rather than only at a singleton information set. We then proceed by analogy: If the players' strate-

[1]Not in the syllabus

gies are to be a Perfect Bayesian equilibrium, they must not only be a Bayesian Nash equilibrium for the entire game but must also constitute a Bayesian Nash equilibrium in every continuation game.

5.1 Perfect Bayesian Equilibrium

Example 5.1. *Player 1 chooses among three actions L, M and R. If Player 1 chooses R then the game ends without a move by Player 2. If Player 1 chooses either L or M, then Player 2 learns that R was not chosen (but not which of L or M was chosen) and chooses between two actions, T and B, after which the game ends. Payoffs are given in the extensive form in Figure 5.1*



		Player 2	
		T	B
Player 1	L	$2, 1$	$0, 0$
	M	$0, 2$	$0, 1$
	R	$1, 3$	$1, 3$

Normal Form of the Game

Figure 5.1: Game with Imperfect Information

Using the normal form representation of this game, we see that there are two pure-strategy Nash equilibria – (L, T) and (R, B). To determine whether these Nash equilibria are subgame-perfect, we use the extensive-form representation to define the game's subgames. Because a subgame is defined to begin at a decision node that is a singleton information set, the entire game in Figure 5.1 is the only subgame. So, both (L, T) and (R, B) are Subgame-Perfect equilibria of this game.

However, (R, B) depends on a noncredible threat: If Player 2 gets the move, then playing T strictly dominates playing B, so Player 1 should not be induced to play R by Player 2's threat to play B if given the move.

One way to strengthen the equilibrium concept so as to rule out the Subgame-perfect Nash equilibrium (R, B) in Figure 5.1 is to impose the following two requirements.

Requirement 1. *At each information set, the player with the move must have a belief about which node in the information set has been reached by the play of the game. For a nonsingleton information set, a belief is a probability distribution over the nodes in the information set; for a singleton information set, the player's belief puts probability one on the single decision node.*

Requirement 2. *Given their beliefs, the players' strategies must be* **sequentially rational**. *That is, at each information set the action taken by the player with the move* *(and the player's subsequent strategy) must be optimal given the player's belief at that information set and the other players' subsequent strategies (where a "subsequent strategy" is a complete plan of action covering every contingency that might arise after the given information set has been reached).*

In Figure 5.1, Requirement 1 implies that if the play of the game reaches Player 2's nonsingleton information set then Player 2 must have a belief about which node has been reached (or, equivalently, about whether Player 1 has played L or M). This belief is represented by the probabilities p and $1-p$ attached to the relevant nodes in the tree, as shown in Figure 5.2.

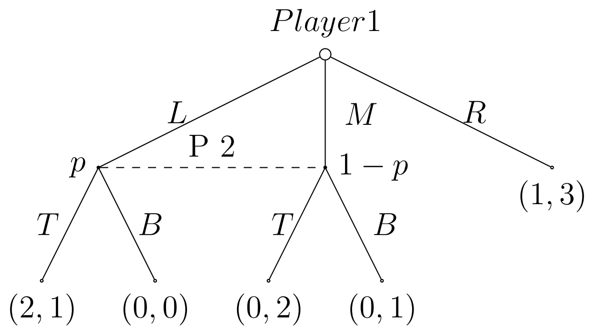


Figure 5.2: Game with Imperfect Information

Given Player 2's belief, the expected payoff from playing B is $p \cdot 0 + (1-p) \cdot 1 = 1-p$, while the expected payoff from playing T is $p \cdot 1 + (1-p) \cdot 2 = 2-p$. Since $2-p > 1-p$ for any value of p, Requirement 2 prevents Player 2 from choosing B. Thus, simply requiring that each player have a belief and act optimally given this belief suffices to eliminate the implausible equilibrium (R, B) in this example.

Requirements 1 and 2 insist that the players have beliefs and act optimally given these beliefs, but not that these beliefs be reasonable. In order to impose further requirements on the players' beliefs, we distinguish between information sets that are *on the equilibrium path* and those that are *off the equilibrium path*.

Definition 5.1. *For a given equilibrium in a given extensive-form game, on information set is on the equilibrium path if it will be reached with positive probability if the game is played according to the equilibrium strategies, and is off the equilibrium path if it is certain not to be reached if the game is played according to the equilibrium strategies (where "equilibrium" can mean Nash, subgame-perfect, Bayesian, or perfect Bayesian equilibrium).*

Requirement 3. *At information sets on the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies.*

In the subgame-perfect Nash equilibrium (L, T) in Figure 5.2, for example, Player 2's belief must be $p = 1$: Given Player 1's equilibrium strategy (namely, L), Player 2 knows which node in the information set has been reached. As a second (hypothetical illustration of Requirement 3, suppose that in Figure 5.2 there were a mixed-strategy equilibrium in which Player 1 plays L with probability q_1, M with probability q_2 and R with probability $1 - q_1 - q_2$. Then Requirement 3 would force Player 2's belief to be $\dfrac{q}{1-q}$.

Requirements 1 through 3 capture the spirit of a Perfect Bayesian equilibrium. The crucial new feature of this equilibrium concept is due to Kreps and Wilson (1982): Beliefs are elevated to the level of importance of strategies in the definition of equilibrium. *Formally an equilibrium no longer consists of just a strategy for each player but now also includes a belief for each player at each information set at which the player has the move.* The advantage of making the players' beliefs explicit in this way is that, just as in earlier chapters we insisted that the players choose credible strategies, we can now also insist that they hold reasonable beliefs, both on the equilibrium path (in Requirement 3) and off the equilibrium path (in Requirement 4, which follows).

Different authors have used different definitions of perfect Bayesian equilibrium. All definitions include Requirements 1 through 3; most also include Requirement 4; some impose further requirements. We take Requirements 1 through 4 to be the definition of Perfect Bayesian equilibrium.

Requirement 4. *At information sets off the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies where possible.*

Definition 5.2. *A Perfect Bayesian equilibrium consists of strategies and beliefs satisfying Requirements 1 through 4.*

Part II

Market Failure

Chapter 6

Overview and The Theory of Second Best

6.1 The Efficiency of Competitive Markets: A Recap

In our analysis of general equilibrium and economic efficiency (done in Microeconomic Theory I), we obtained two remarkable results. First, we have shown that for any initial allocation of resources, a competitive process of exchange among individuals, whether through exchange, input markets, or output markets, will lead to a Pareto efficient outcome. The first theorem of welfare economics tells us that a competitive system, building on the self-interested goals of consumers and producers and on the ability of market prices to convey information to both parties, will achieve a Pareto efficient allocation of resources.

Second, we have shown that with indifference curves that are convex, any efficient allocation of resources can be achieved by a competitive process with a suitable redistribution of those resources. Of course, there may be many Pareto efficient outcomes. But the second theorem of welfare economics tells us that under certain (admittedly ideal) conditions, issues of equity and efficiency can be treated distinctly from one another. If we are willing to put equity issues aside, then we know that there is a competitive equilibrium that maximizes consumer and producer surplus, i.e., is economically efficient.

Both theorems of welfare economics depend crucially on the assumption that markets are competitive. Unfortunately, neither of these results necessarily holds when, for some reason, markets are no longer competitive. In this part, we will discuss ways in which markets fail and what government can do about it. Before proceeding, however, it is essential to review our understanding of the workings of the competitive process. We thus list the conditions required for economic efficiency in exchange, in input markets, and in output markets.

1. *Efficiency in exchange:* All allocations must lie on the exchange contract curve so that every consumer's marginal rate of substitution of good i (say food) for good j

(say clothing) is the same:[1]

$$MRS^i_{FC} = MRS^j_{FC}$$

A competitive market achieves this efficient outcome because, for consumers, the tangency of the budget line and the highest attainable indifference curve ensure that:

$$MRS^i_{FC} = \frac{p_f}{p_c} = MRS^j_{FC}.$$

2. *Efficiency in the use of inputs in production:* Every producer's marginal rate of technical substitution of labour for capital is equal in the production of both goods:

$$MRTS^F_{LK} = MRTS^C_{LK}$$

A competitive market achieves this technically efficient outcome because each producer maximizes profit by choosing labour and capital inputs so that the ratio of the input prices is equal to the marginal rate of technical substitution:

$$MRTS^F_{LK} = \frac{w}{r} = MRTS^C_{LK}.$$

3. *Efficiency in the output market:* The mix of outputs must be chosen so that the marginal rate of transformation between outputs is equal to consumers' marginal rates of substitution:

$$MRT_{FC} = MRS_{FC} \text{ (for all consumers)}$$

A competitive market achieves this efficient outcome because profit-maximizing producers increase their output to the point at which marginal cost equals price:

$$p_F = MC_F, \; p_C = MC_c$$

As a result,

$$MRT_{FC} = \frac{MC_F}{MC_C} = \frac{p_F}{p_C}.$$

But consumers maximize their satisfaction in competitive markets only if

$$\frac{p_F}{p_c} = MRS_{FC} \text{ (for all consumers)}$$

Therefore,

$$MRS_{FC} = MRT_{FC}.$$

and the output efficiency conditions are satisfied. Thus efficiency requires that

[1] Superscripts denote consumers and subscripts denote goods.

goods be produced in combinations and at costs that match people's willingness to pay for them.

6.2 Why Markets Fail?

Competitive markets fail for four basic reasons: *externalities*, *public goods*, *market power*, and *incomplete information*. We will discuss each in turn.

1. **Externalities**, as we will see, arise when one consumer's utility and/or. firm's production set get affected by the activities of other economic agent. The word externality is used because the effects on others (whether benefits or costs) are external to the market.

   Suppose, for example, that a steel plant dumps effluent in a river, thus making a recreation site downstream unsuitable for swimming or fishing. There is an externality because the steel producer does not bear the true cost of waste water and so uses too much waste water to produce its steel. This externality causes an input inefficiency. If this externality prevails throughout the industry, the price of steel (which is equal to the marginal cost of production) will be lower than if the cost of production reflected the effluent cost. As a result, too much steel will be produced, and there will be an output inefficiency.

   We will discuss externalities and ways to deal with them in Chapter 7.

2. **Public Goods** A public good is a good which is jointly consumed by persons. Since it is not possible to exclude persons from consumption of a jointly consumed good. there may not be any incentive for individual to pay for the good. This cause of market failure is known as the free-rider problem (see Chapter 8).

3. **Market Power** In many cases and for different reasons there are sufficiently few persons on one side of the market and they know that their actions may alter market prices. Markets that are organized as oligopolies or in which there is a monopsony on the buying side are considered in this category, in addition to the usual monopoly case. The essential effects of all such market is that the marginal revenue (or marginal expense in the case of monopsony) differs from the market price. Consequently, relative prices no longer reflect relative marginal costs, which in turn means that the price system does not convey the information necessary to ensure efficiency. We address this in Chapter 8.

4. **Incomplete Information** In Chapters 9 and 10, we consider situations in which an asymmetry of information exists among market participants. The complete markets assumption of the first welfare theorem implicitly requires that the characteristics of traded commodities be observable by all market participants because, without

this observability, distinct markets cannot exist for commodities that have different characteristics.

Chapter 10 focuses on the case in which asymmetric information exists between agents at the time of contracting. Our discussion highlights several phenomena – *adverse selection, signaling, and screening* – that can arise as a result of this informational imperfection, and the welfare loss that it causes.

Chapter 11 in contrast, investigates the case of post-contractual asymmetric information, a problem that leads us to the study of the principal-agent model. Here, too, the presence of asymmetric information prevents trade of all relevant commodities and can lead market outcomes to be Pareto inefficient.

6.3 Market Failure and the Theory of Second-Best

It will be clearer that the issues discussed above are relevant to any economic system. One may argue that a competitive price system could still be advocated for those sectors of the economy in which these impediments do not exist. By using the price system in these sectors, the Pareto rules for allocational efficiency would be satisfied there; this would be a second best solution to the problem. But unfortunately *unless one corrects all of these distortions/market failures* so to bring about a best Pareto efficient allocation, this approach to find a second-best solution is *not correct.*

Lipsey and Lancaster (1956) addressed this problem in a general framework of policy optimization and reached an essentially negative conclusion. Their general theory for the *second-best optimum* states that in general, if one or more of the Pareto conditions are unattainable, then others, still attainable, are not desirable. From this theorem it follows that there is *no a priori way to rank different states of the world on the basis of the remaining Pareto optimality conditions*. It is not true that having more (but not all) of the efficiency conditions satisfied is necessarily superior to having fewer hold.

The central idea underlying the general theorem of the second-best can be explained graphically. Suppose that a society's production possibilities for the two goods are represented by the frontier FF' in Figure 6.1 and that preferences are given by the indifference curves. There in a constraint in the economy that makes the actual optimum point B unattainable. This constraint is represented by the line CC'. That is, combination of the goods to the top right of the line CC' cannot be achieved because of the constraint. Society wishes to maximize welfare represented by the indifference curves subject to the constraint CC'. From the figure it is clear that the welfare maximizing point need not be on the frontier FF'. This demonstrates the principle negative result of the theory of the second-best: *If all the conditions of Pareto efficiency cannot be fulfilled, it is not true that to satisfy some of them is the best policy.* In such a case an optimum situation can be attained only by departing from the remaining Paretian conditions. The optimum situation

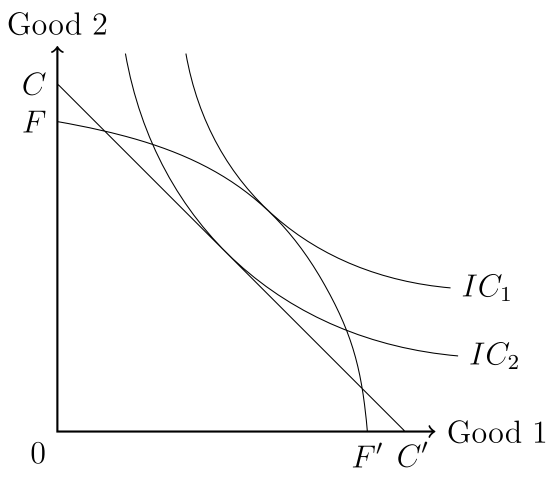


Figure 6.1: The Theory of Second Best

finally achieved maybe regarded as a second best optimum because it is achieved subject to a constraint which, by definition, does not allow attainment of a Paretian optimum.

Appendix[2]

To state the problem technically, suppose we need to maximize an objective function $f(x_1, ..., x_n)$ subject to the feasibility constraint $g(x_1, ..., x_n) = 0$. Let us consider the Lagrangian

$$\mathcal{L} = f(x_1, ..., x_n) + \lambda g(x_1, ..., x_n),$$

where λ is the Lagrange multiplier. Then differentiating $\mathcal{L}$ with respect to x_1, x_2, ..., x_n and λ, and setting the partial derivatives to zero, we get the first order conditions

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial x_1} = f_1(x_1, ..., x_n) + \lambda g_1(x_1, ..., x_n) = 0, \\ &\vdots \\ &\frac{\partial \mathcal{L}}{\partial x_n} = f_n(x_1, ..., x_n) + \lambda g_n(x_1, ..., x_n) = 0, \\ \text{and} \quad &\frac{\partial \mathcal{L}}{\partial \lambda} = g(x_1, ..., x_n) = 0, \end{aligned}$$

where $f_i(x_1, ..., x_n)$ and $g_i(x_1, ..., x_n)$ are the partial derivatives of $f(\cdot)$ and $g(\cdot)$ with respect to x_i, where $i = 1, ..., n$. We can rewrite the above conditions as:

[2] Not in syllabus

$$\begin{aligned}
&\frac{f_1(x_1,...,x_n)}{f_n(x_1,...,x_n)} = \frac{g_1(x_1,...,x_n)}{g_1(x_1,...,x_n)},\\
&\vdots\\
&\frac{f_{n-1}(x_1,...,x_n)}{f_n(x_1,...,x_n)} = \frac{g_{n-1}(x_1,...,x_n)}{g_1(x_1,...,x_n)},\\
\text{and}\quad & g(x_1,...,x_n) = 0,
\end{aligned}$$

We have a second best situation if at least one of the first $(n-1)$ equalities does not hold. Observe that equality $g(x_1,...,x_n) = 0$ has to hold because it is given a priori. But the other $(n-1)$ equalities are derived as optimizing conditions. For instance, if we have $\frac{f_1(x_1,...,x_n)}{f_n(x_1,...,x_n)} = k[\frac{g_1(x_1,...,x_n)}{g_1(x_1,...,x_n)}]$, where $k \neq 1$, then a second best situation arises.

Suppose, for example, that f is the social welfare function in a two-person (A and B) two-good (1 and 2) community, then f is a function of $u^i(x^1) = u^i(x_1^i, x_2^i)$ and $u^j(x^1) = u^j(x_1^j, x_2^j)$ where $u^i(\cdot)$ is person i's utility function, x_1^i is person i's consumption of good 1, and so on. The constraint $g(\cdot)$ becomes the production possibility frontier $T(\omega_1, \omega_2) = 0$, where ω_1 is the total amount of good 1, that is $\omega_1 = \omega_1^A + \omega_2^B$. Similarly, ω_2 denotes the total amount of good 2, that is $\omega_2 = \omega_2^A + \omega_2^B$. Thus, we are considering the problem of social welfare.

Recall, that the conditions required for Pareto efficient allocation or the the first-best conditions are

$$\frac{\frac{\partial u^A(x_1^A, x_2^A)}{\partial x_1^A}}{\frac{\partial u^A(x_1^A, x_2^A)}{\partial x_2^A}} = \frac{\frac{\partial T}{\partial \omega_1}}{\frac{\partial T}{\partial \omega_2}}$$

$$\frac{\frac{\partial u^B(x_1^B, x_2^B)}{\partial x_1^B}}{\frac{\partial u^B(x_1^B, x_2^B)}{\partial x_2^B}} = \frac{\frac{\partial T}{\partial \omega_1}}{\frac{\partial T}{\partial \omega_2}}$$

$$T(\omega_1, \omega_2) = 0.$$

A second-best situation arises here if at least one of the first two equalities does not hold. There is no doubt that the Lipsey-Lancaster theorem has given a negative implication in the sense that undermines the Pareto efficiency principle stating that if any one of the Pareto efficiency conditions is not met, then satisfying the remaining conditions is not necessarily desirable. It means that the second-best optimality conditions are much more difficult than the simple Pareto optimality conditions. However the second-best framework is essential for treating economic problems in the real world.

Chapter 7

Externality

With this chapter, we begin our study of market failures: situations in which some of the assumptions of the welfare theorems do not hold and in which, as a consequence, market equilibria cannot be relied on to yield Pareto optimal outcomes.

Till now, we assumed that the preferences of a consumer were defined solely over the set of goods that she might herself decide to consume. Similarly, the production of a firm depended only on its own input choices. In reality, however, a consumer or firm may in some circumstances be *directly* affected by the actions of other agents in the economy; that is, there may be external effects from the activities of other consumers or firms. For example, the consumption by consumer i's neighbour of loud music at three in the morning may prevent her from sleeping. Likewise, a fishery's catch may be impaired by the discharges of an upstream chemical plant. Incorporating these concerns into our preference and technology formalism is, in principle, a simple matter: We need only define an agent's preferences or production set over both her own actions and those of the agent creating the external effect. But the effect on market equilibrium is significant: In general, when external effects are present, competitive equilibria are *not Pareto optimal*.

Definition 7.1. *An externality is present whenever the well-being of a consumer or the production possibilities of a firm are* **directly affected** *by the actions of another agent in the economy.*

Note the definition carefully, it contains a subtle point that has been a source of some confusion. When we say *directly* we mean to exclude any effects that are mediated by prices. That is, an externality is present if, say, a fishery's productivity is affected by the emissions from a nearby oil refinery, but not simply because the fishery's profitability is affected by the price of oil (which, in turn, is to some degree affected by the oil refinery's output of oil). The latter type of effect (referred to as a pecuniary externality by Viner (1931)) is present in any competitive market but it creates no inefficiency. Indeed, with price-taking behavior, the market is precisely the mechanism that guarantees a Pareto optimal outcome. This suggests that the presence of an externality is not merely a technological phenomenon but also a function of the set of markets in existence. We return to this point later in the section 7.2.4. An externality favorable to the recipient is usually called a *positive externality*, and conversely for a negative externality.

Externalities can take place between consumers, between firms and consumers, and also between firms. If a consumer's preference is directly affected by goods consumed by another consumer or a firm's production, then we say that *consumption externality* exists. Many of the impressive and exciting sights of a large city are the skyscrapers built by private firms. This is an example of a positive consumption externality. Another example of an externality of this type can be the benefit that a person gets from a better-lit footpath when his neighbour installs a light outside the gates of the neighbour's house. But suppose his neighbour is a motorcycle enthusiast and does not believe in using a muffler. Then it is probable that his comings and goings at all hours will cause noise. This is an example of negative consumption externality.

When the production of one firm gets affected by the choice of a consumer or the production of another firm, we say that production externalities exist. The pollination of an apple orchard by a honey producer's bees improves productivity of the apple industry. Conversely, since the bees feed on apple blossoms, an increase in apple production will improve productivity of the honey industry. Thus, in this classic case we have mutual positive production externalities. Next, suppose two firms are located on the same river. Firm A, the upstream firm, dumps its wastes into the river, while firm B, the downstream firm,uses river water for processing its output. If firm A increases its output (hence wastes, firm B's production suffers. To produce the same output with dirtier water firm B uses more labor, more chemicals, etc. (More specifically we may assume that firm B is a fishery so that as the amount of wastes dumped into the river increases, its output gets adversely affected to a higher extent.) This is an example of negative production externality – firm A's production negatively affects firm B's production.

In all the above situations the problem is that there are no markets for externalities, whether they are in consumption or in production, whether they are positive or negative: That is, there is no market for noise created by motorcycles or wastes dumped into the river and so on. In the case of negative externalities the decision maker does not know how much his decision really costs. In the case of positive externalities, the decision maker does not know how much his decision really helps.

In the remainder of this chapter, we explore the implications of external effects for competitive equilibria, we will see that the link between Pareto efficiency and competition is broken. Then we will see how appropriate corrective mechanisms can achieve Pareto efficiency.

7.1 Production Externality

7.1.1 Nonoptimality of the Competitive Outcome

To illustrate production externalities analytically, let us consider the two firm example again, where the upstream firm A produces some amount of x_1^A and also produces a certain amount of wastes, s, which it dumps into the river. The downstream firm (B) gets adversely affected by firm A's wastes.

We denote firm A's cost function by $c^A(x_1^A, s)$ and firm B's cost function by $c^B(x_2^B, s)$, where x_2^B is firm B's production of its good, 2, and s is the amount of wastes produced and dumped into the river by firm A (We assume here that the input markets that firms A and B face are perfectly competitive. Note that s directly affects firm B's cost of producing good 2. We assume that firm B's cost of production increases as the wastes dumped into the river increase, $\dfrac{\partial c^B(x_2^B, s)}{\partial s} > 0$, and that increased wastes do not increase firm A's cost of production; $\dfrac{\partial c^A(x_1^A, s)}{\partial s} \leq 0$.
Firm A's profit maximization problem is

$$\max_{x_1^A, s} p_1 x_1^A - c^A(x_1^A, s)$$

and firm B's profit maximization problem is

$$\max_{x_2^B} p_2 x_2^B - c^B(x_2^B, s),$$

where p_1 and p_2 are competitive prices of goods 1 and 2 respectively. Since firm B does not have any control over the amount of wastes dumped into the river, it cannot regard s as an optimizing variable.
The profit maximizing conditions for firm A are

$$\frac{\partial c^A(x_1^A, s)}{\partial x_1^A} = p_1, \tag{7.1.1}$$

$$\frac{\partial c^A(x_1^A, s)}{\partial s} = 0. \tag{7.1.2}$$

For firm B the only profit maximizing condition turns out to be

$$\frac{\partial c^B(x_2^B, s)}{\partial x_2^B} = p_2. \tag{7.1.3}$$

According to these conditions, at profit maximizing point the price of each good should be equal to its marginal cost. Since the price of wastes dumped into the river is zero for firm A, the profit maximizing condition requires that the firm will dump wastes until the

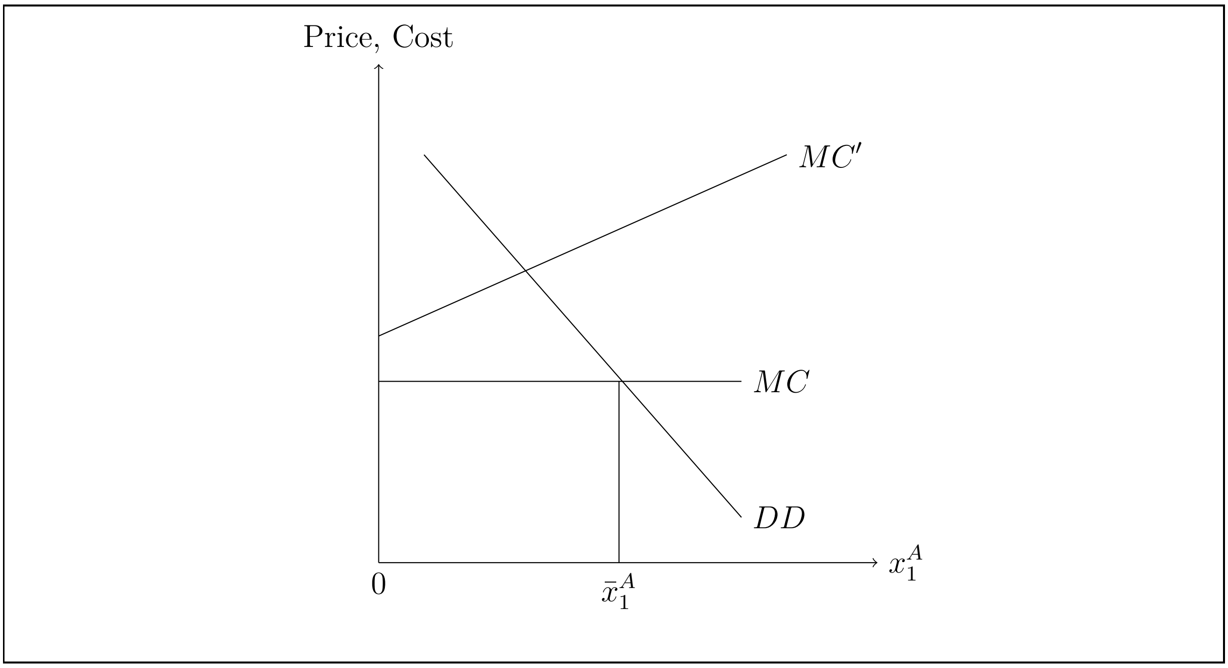


Figure 7.1: The Equilibrium Level Under Negative Externality

cost of an additional unit is zero.

We can easily recognize the presence of externality in the model formulated above. Firm B cares about wastes but has no control over it. When firm A increases its output, it increases the cost of production for firm B. Because this cost is external to firm A, it ignores the cost when making its decision of how much to produce. The increase in cost of production of firm B associated with an increase in wastes dumped by firm A is a part of the social cost of production of firm A's output. In general, firm A will dump too much waste since it ignores the impact of that on firm B and good 1 will be over-produced.

We may illustrate the situation graphically (see figure 7.1). The demand for firm A's output is given by the curve DD and MC denotes A's marginal cost curve. The curve MC' records the social marginal cost of firm A. MC' differs from MC by the amount of extra costs that firm A imposes on others (here only firm B) in the society.

The normal workings of the market will cause output level $\bar{x}_1^A$, to be produced, since at this output market price is equal to the private marginal cost MC (equation (7.1.4)). Since the other profit maximizing condition says that the marginal cost of wastes dumped is zero, firm A ignores the difference between MC' and MC.

7.1.2 Resolution of Production Externalities: Traditional Ways

There is a number of potential solutions to the allocational problems posed above. We will first examine two "traditional" solutions: Internalization, and taxation of costs. Then we will see that in some circumstances externalities can be accommodated by the normal workings of the market and traditional solutions may be unnecessary.

7.1.2.1 Merger and Internalization

A traditional cure for the allocational distortions caused by production externalities would be merging of two firms. If there was a single firm operating both the plants of A and B, it would recognize the detrimental effect that firm A has on the production function of firm B. Then there will be no externality! We say that the externality has been internalized as a result of the merger.

Attempts to internalize externalities in productions are quite common. It is often the case that an organization will expand in size with the purpose of encompassing all the spillover effects of its activities. Firms, for example, may merge to capture external benefits. Recreational site developers (ski areas, golf courses, resort hotels) often operate the service industries (hotels, gas stations, shops) near their projects. In this way they are able to internalize the positive benefits that such developments provide to the service industries. Similarly, in the case of the apple orchard and bee keeper example mentioned earlier, it may be possible that more profits could be earned if one or both firms coordinated their activities either by mutual agreement or by sale of one of the firms to the other. Therefore market, more precisely, profit maximization may encourage/signal internalization.

Let us now go back to the production externality problem discussed in the previous section. Before the merger, each firm had the right to produce whatever amount of output it wanted, regardless of what the other firm did. After the merger, the combined firm has the right to control the production of outputs of goods 1 and 2.
The merged firm's profit maximization problem is

$$\max_{x_1^A, x_2^B, s} p_1 x_1^A + p_2 x_2^B - c^A(x_1^A, s) - c^B(x_2^B, s).$$

The optimality conditions turn out to be

$$\frac{\partial c^A(x_1^A, s)}{\partial x_1^A} = p_1, \tag{7.1.4}$$

$$\frac{\partial c^B(x_2^B, s)}{\partial x_2^B} = p_2. \qquad \frac{\partial c^A(x_1^A, s)}{\partial s} + \frac{\partial c^B(x_2^B, s)}{\partial s} = 0. \tag{7.1.5}$$

The last condition says that the merged firm will take into account the effect of wastes dumped on the marginal costs of firms A and B. That is, under merger firm A takes the social cost of its production into account. In other words, the merged firm would now pay the full social marginal cost of production of good 1 because it also produces good 2. Consequently, in figure 7.1 the marginal cost curve for producing good 1 becomes MC'. This in turn shows that the profit maximizing output of good 1 becomes $\hat{x}_1^A$. This is the optimal output level from a social point of view. At this output level the marginal benefit of good 1's production (what people are willing to pay for the good) is exactly equal to the marginal social cost. Thus, as indicated earlier, under normal workings of

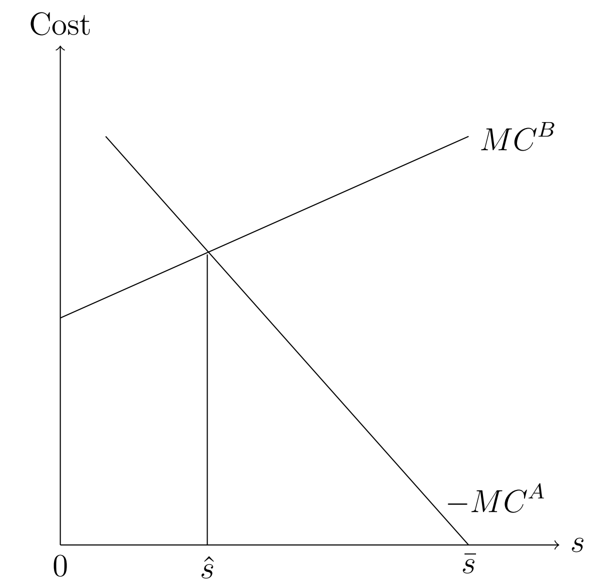


Figure 7.2: Cost of Externality: Equilibrium and Optimal Level of Waste

the market good 1 will be over-produced ($\hat{x}_1^A < \bar{x}_1^A$).
We can rewrite equation (7.1.5) as

$$-\frac{\partial c^A(x_1^A, s)}{\partial s} = \frac{\partial c^B(x_2^B, s)}{\partial s} > 0 \tag{7.1.6}$$

The right hand side of equation (7.1.6), which is denoted by MC^B, measures the (positive) marginal cost to the downstream firm B of more wastes. Therefore, the left hand side, which we denote by $-MC^A$ gives the marginal cost of firm A from dumping more wastes. The upstream firm independently dumps wastes up to the point where its marginal cost from dumping more waste is zero. But under merger, firm A dumps wastes up to the point where the effect of marginal increase in wastes is equal to the marginal social cost, which is positive. This positive marginal social cost counts the impact of wastes on the costs of both firms. Hence the merged firm will dump less wastes than the independent firm A. In other words, when the true social cost of externality involved in the production of firm A's output is taken into account, the optimal dumping of wastes takes place and this optimal amount is smaller than the amount dumped by independent firm A. Since the optimal level of wastes is determined by equation (7.1.5), the intersection of the two marginal costs on the left and right hand sides of equation (7.1.6) also gives the optimal waste level (see figure 7.2). As stated earlier, at the efficient level of wastes dumped, the amount that firm A is willing to pay for an extra unit of waste should equal the social cost generated by dumped wastes. Under the corrective mechanism, which ensures equality between private and social costs, the free market will determine a Pareto efficient amount of output of each good.

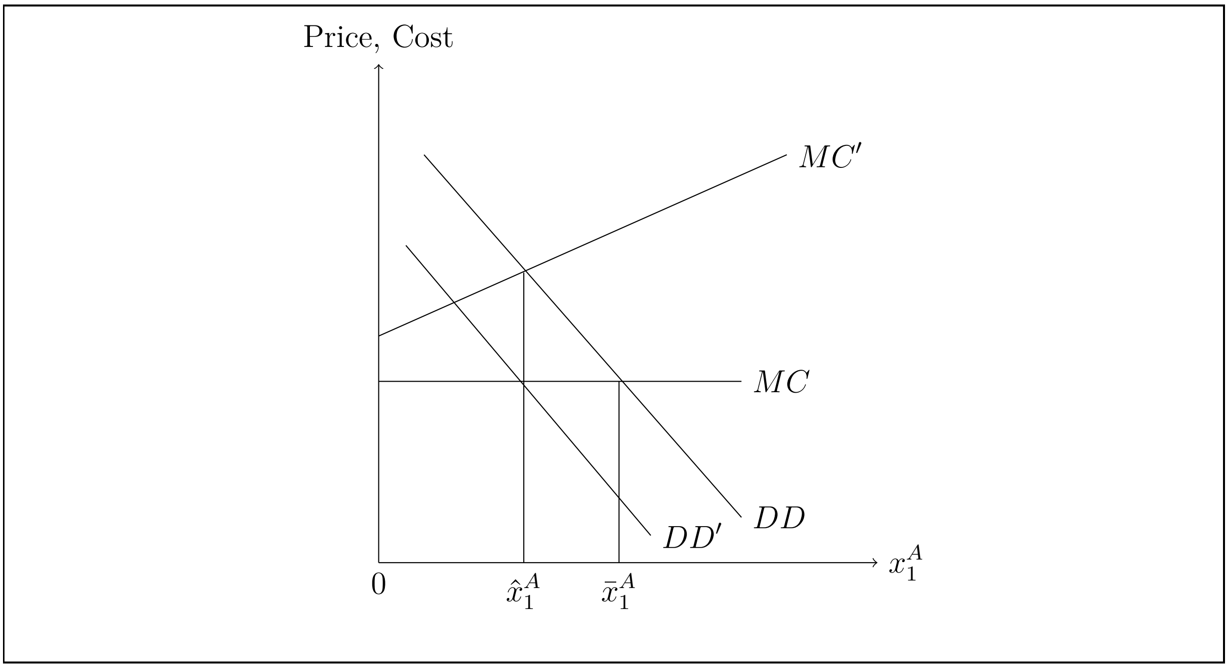


Figure 7.3: Cost of Externality and Pigouvian Tax

7.1.2.2 Pigouvian Taxation

The government could impose a suitable tax on the firm generating externality. The imposed tax would cause the output of firm A, good 1, to be cut back and would cause inputs to be shifted out of production of good 1. This classic remedy of the externality problem was suggested by Pigou (1920).

The taxation problem can be illustrated in terms of figure 7.3. A specific tax of amount t would cause the demand curve DD to shift to DD'. With this new demand curve, the private profit maximizing output will be $\hat{x}_1^A$, the marginal external damage done by producing good 1 is given by the distance ad, which is the amount t. By taxing the good, its effective demand is reduced and resources are shifted out of its production.

To discuss the solution analytically, note that under taxation the profit maximization problem of firm A becomes

$$\max_{x_1^A, s} p_1 x_1^A - c^A(x_1^A, s) - ts$$

which gives the following optimality conditions

$$\frac{\partial c^A(x_1^A, s)}{\partial x_1^A} = p_1, \tag{7.1.7}$$

$$-\frac{\partial c^A(x_1^A, s)}{\partial s} = t. \tag{7.1.8}$$

Comparing these two equations to equation (7.1.6) , we observe that letting

$$t = \frac{\partial c^B(x_2^B, s)}{\partial s}$$

will make these conditions the same as the conditions that characterize the optimal level of wastes.

The central problem with Pigouvian taxes is that regulators require empirical information about the optimal level of wastes in order to impose the tax. But once the optimal level of wastes is known, firm A may be asked to dump exactly that much.

7.1.3 Resolution of Production Externalities: Property Rights and the Coase Theory

Another approach to the externality problem aims at a less intrusive form of intervention, merely seeking to insure that conditions are met for the parties to themselves reach an optimal agreement on the level of the externality.

This requires a *well-defined* system of *property rights* that excludes persons/firms from using a good or factor for which they have not paid. Property right establishes the legal owner of a resource and specifies the way in which the resource may be used. It allows agents to exclude themselves from consuming an unwanted commodity or factor without compensation.

Two major types of property specifications are common property and private property. Common property is solely owned by the society at large. For such properties it is prohibitively costly or otherwise infeasible to assign and/or enforce property rights to individuals. Private property, on the other hand, is directly owned by individuals who have, within prevailing legal structures, some say over how it is used. Private property may be either exchangeable or non-exchangeable, depending on whether the good in question can be traded or not. (An interesting example of a non exchangeable property is an individual's vote, a private good provided by the state.)

When we have well-defined property rights in the good involving externality, the concerned parties can trade from their initial endowments to a Pareto efficient state. This result is known as the Coase theorem, after the economist who proposed it (Coase, 1960). Formally, the Coase theorem states the following: If trade of the externality can occur, then bargaining will lead to an efficient outcome no matter how property rights are allocated.

We now demonstrate the Coase theorem for our two-firm example. Suppose, initially, that firm A owns the river. Then it must impute some cost of this ownership into its cost function. This cost of river ownership is determined by the next-best alternative use of the river. In this case only firm B has some alternative use for the river (to keep it clean), and the amount that this firm would be willing to pay to keep the river clean is equal to the external damage done by the wastes dumped. Therefore, if firm A calculates its cost correctly, its marginal cost curve (including the cost of river ownership) will be MC' in figure 7.1. Firm A will therefore produce $\hat{x}_1^A$ and sell the remaining rights of river use to firm B for a fee of some amount between *abd* (A's loss from producing $\hat{x}_1^A$ rather than $\bar{x}_1^A$) and *abcd* (the maximum amount that B would be willing to pay to avoid output of good 1 increasing from $\hat{x}_1^A$ to $\bar{x}_1^A$).

A similar result emerges if firm B owns the river. In this case firm A would be willing to pay an amount up to the total profit it earns to achieve the right to dump into the river. Firm B will accept a payment as long it exceeds the cost resulting from dumping of wastes. The ultimate point of bargaining will be for firm A to offer a payment to firm B for dumping wastes associated with the output level $\hat{x}_1^A$ into the river. Firm B will not sell the rights to dump any more waste because what firm A will pay in exchange falls short of the costs associated with dumped wastes. Therefore, an efficient point is reached through bargaining between the two firms.

It is interesting to note that in both situations some production of good 1 (hence dumping of wastes) takes place. Having no output of good 1 (and no waste dumped) becomes inefficient. Thus, there is some optimal level of pollution and this level is determined thorough bargaining between concerned parties.

Now we demonstrate the Coase theorem analytically. Suppose firm B has the right to clean water and wishes to sell the right to firm A to allow dumping of wastes. Let r be the price per unit of waste dumped and s be the amount of wastes that firm A dumps. (Note that when firm A was acting independently, it used to treat r as zero.) Then the profit maximization problems of firms A and B are

$$\max_{x_1^A, s} p_1 x_1^A - rs - c^A(x_1^A, s),$$
$$\max_{x_2^B, s} p_2 x_2^B + rs - c^B(x_2^B, s).$$

Since firm A buys the right to dump s units of wastes into the river, the term rs enters with a negative sign in its profit expression. Conversely, because firm B earns a revenue of rs by selling the right, rs appears with a positive sign in B's expression.
The profit maximizing conditions are

$$\frac{\partial c^A(x_1^A, s)}{\partial x_1^A} = p_1, \tag{7.1.9}$$
$$-\frac{\partial c^A(x_1^A, s)}{\partial s} = r, \tag{7.1.10}$$
$$\frac{\partial c^B(x_2^B, s)}{\partial x_2^B} = p_2, \tag{7.1.11}$$
$$\frac{\partial c^B(x_2^B, s)}{\partial s} = r. \tag{7.1.12}$$

The conditions of equations (7.1.10) and (7.1.12) show that the marginal cost to firm A of reducing waste dumping should be equal to the marginal benefit to firm B of that reduction. Equality between these two marginal costs is necessary for production of the optimal pollution level. This is the same condition we had in equation (7.1.6) . We can also interpret this equality condition as follows: If the price of pollution is adjusted until the demand for wastes equals the supply of wastes, we will have an efficient level of wastes dumped, just as with any other good.

Conversely, suppose that firm A has the right to dump waste and firm B will be willing to pay firm A for reduction of wastes. Now suppose firm A dumps wastes up to certain amount, say, s_m. But firm B will pay firm A to reduce this. The profit maximization problems now become

$$\max_{x_1^A, s} p_1 x_1^A + r(s_m - s) - c^A(x_1^A, s),$$
$$\max_{x_2^B, s} p_2 x_2^B - r(s_m - s) - c^B(x_2^B, s).$$

The optimality conditions turn out to be,

$$\frac{\partial c^A(x_1^A, s)}{\partial x_1^A} = p_1, \tag{7.1.13}$$
$$-\frac{\partial c^A(x_1^A, s)}{\partial s} = r, \tag{7.1.14}$$
$$\frac{\partial c^B(x_2^B, s)}{\partial x_2^B} = p_2, \tag{7.1.15}$$
$$\frac{\partial c^B(x_2^B, s)}{\partial s} = r. \tag{7.1.16}$$

which are the same as the conditions of equations (7.1.11) to (7.1.12). *Thus, the optimal amount of dumping is independent of who owns the river.*

It should be noted that the Coase theorem depends crucially on the assumption that bargaining is costless. If there are costs associated with bargains, then we will have to compare those costs to the potential gains resulting from bargaining. If the gains exceed the bargaining costs, then only will the Coase result hold.

7.2 Consumption Externalities: Example I

Suppose there are L commodities and market prices are $(p_1, ..., p_L)$. We consider 2 consumers in this market. Both the consumers are too small to affect the prices, so they take the prices as given. Let consumer i's wealth at this price vector be w^i.

Each consumer ($i = \{1, 2\}$) has preferences not only over her consumption of L traded goods $(x_1^i, ..., x_L^i)$ but also over some action $h \in \mathbb{R}_+$ taken by *consumer 1*. Thus, consumer

i's (differentiable) utility function takes the form $u^i(x_1^i, ..., x_L^i, h)$, and we assume that

$$\frac{\partial u^2(x_1^2, ...x_L^2, h)}{\partial h} \neq 0$$

Because consumer 1's choice of h affects consumer 2's well-being, it generates an externality.

Let $v^i(p, w^i, h)$ denote a derived utility function over the level h assuming optimal commodity purchases by consumer i at prices $p \in R^L$ and wealth w^i:

$$v^i(p, w^i, h) = \max_{x^i \geq 0} u^i(x^i, h)$$
$$\text{subject to } \; p \cdot x^i \leq w^i.$$

Let us also assume that the consumer's utility function takes a quasilinear form such that

$$v^i(p, w^i, h) = \phi^i(p, h) + w^i.$$

Since the prices of the L traded goods are assumed to be unaffected by any of the changes we are considering, we shall suppress the price vector p and simply write $\phi^i(h)$. Let $\phi^i(h)$ be twice differentiable with $\phi^{i''}(\cdot) < 0$.

7.2.1 Nonoptimality of the Competitive Outcome

Suppose that we are at a competitive equilibrium in which commodity prices are p. That is, at the equilibrium position, each of the two consumers maximizes her utility limited only by her wealth and the prices p of the traded goods. It must therefore be the case that consumer 1 chooses her level of $h \geq 0$ to maximize $\phi^1(h)$. Thus, the equilibrium level of h, denoted by h^*, satisfies the necessary and sufficient first-order condition $\phi^{1'}(h) = 0$.[1]

In contrast, in any Pareto optimal allocation, the optimal level of h, denoted by h^o, must maximize the joint surplus of the two consumers, and so must solve

$$\max_{h \geq 0} \phi^1(h) + \phi^2(h).$$

This problem gives us the necessary and sufficient first-order condition for h^o:

$$\phi^{1'}(h^o) = -\phi^{2'}(h^o)$$

When external effects are present, so that $\phi^{2'}(h) \neq 0$ at all h, the equilibrium level of h is not optimal.

If $\phi^{2'}(h) < 0$, so that h generates a negative externality, then we have $\phi^{1'}(h^o) = -\phi^{2'}(h^o) > 0$. Now since $\phi^{1'}(\cdot)$ is decreasing and $\phi^{1'}(h^*) = 0$, this implies that $h^* > h^o$. In

[1] Observe given our assumption that $\phi^{(}\cdot)$ is concave, S.O.C. is automatically satisfied.

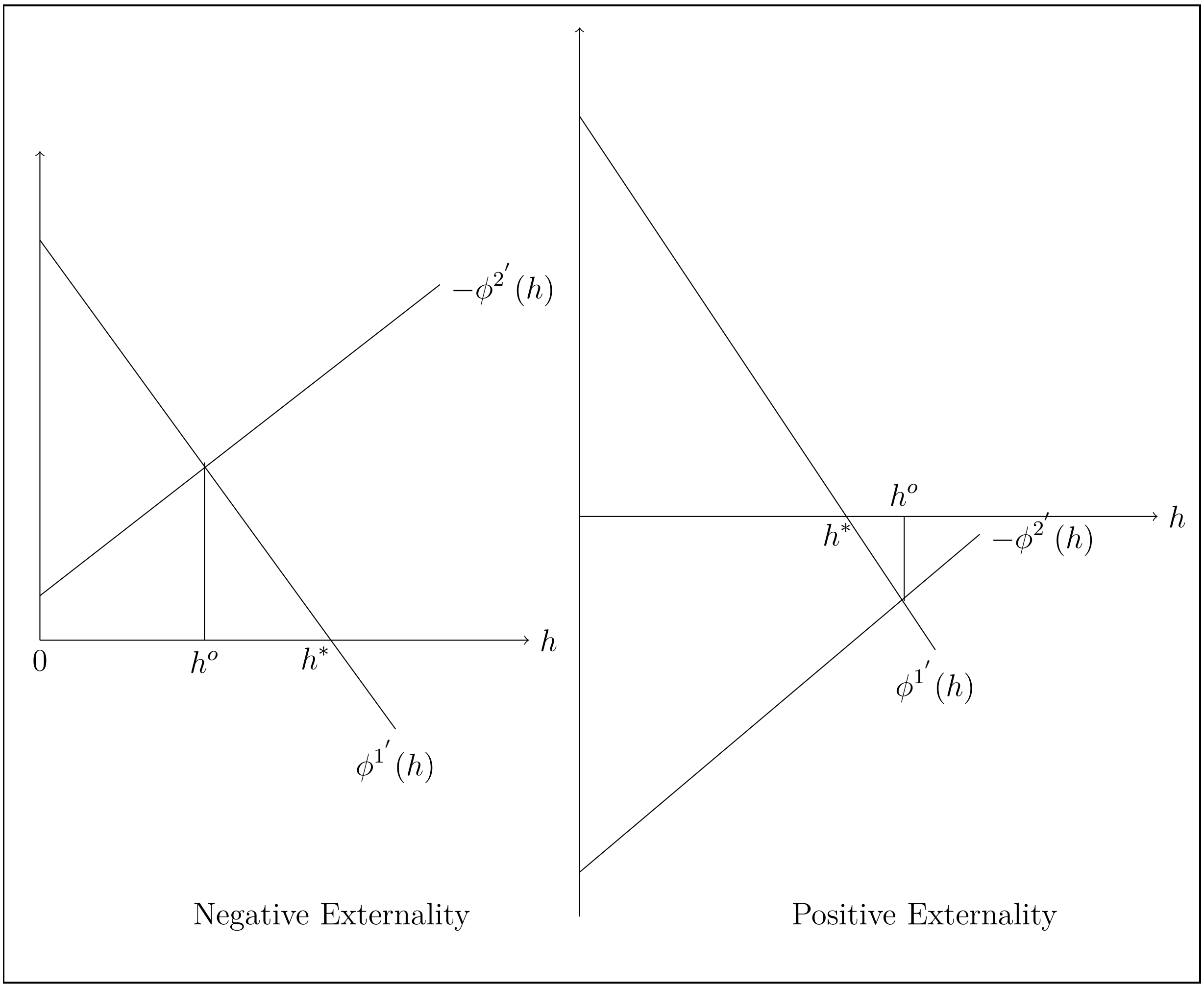


Figure 7.4: The Equilibrium (h^*) and Optimal (h^o) Levels

contrast, when $\phi^{2'}(h) < 0$, h represents a positive externality, and $\phi^{1'}(h^o) = -\phi^{2'}(h^o) < 0$ implies that $h^* < h^o$.

Figure 7.4 depicts these.

Note that here also optimality does not usually entail the complete elimination of a negative externality. Rather, the externality's level is adjusted to the point where the *marginal benefit to consumer 1 of an additional unit of the externality-generating activity,* $\phi^{1'}(h^o)$*, equals its marginal cost to consumer 2,* $-\phi^{2'}(h^o)$.

7.2.2 Traditional Solutions to the Externality Problem

Having identified the inefficiency of the competitive market outcome in the presence of an externality, we now consider three possible solutions to the problem. We first look at government-implemented quotas and taxes, and, then analyze the possibility that an efficient outcome can be achieved in a much less intrusive manner by simply fostering bargaining between the consumers over the extent of the externality (as we have already seen).

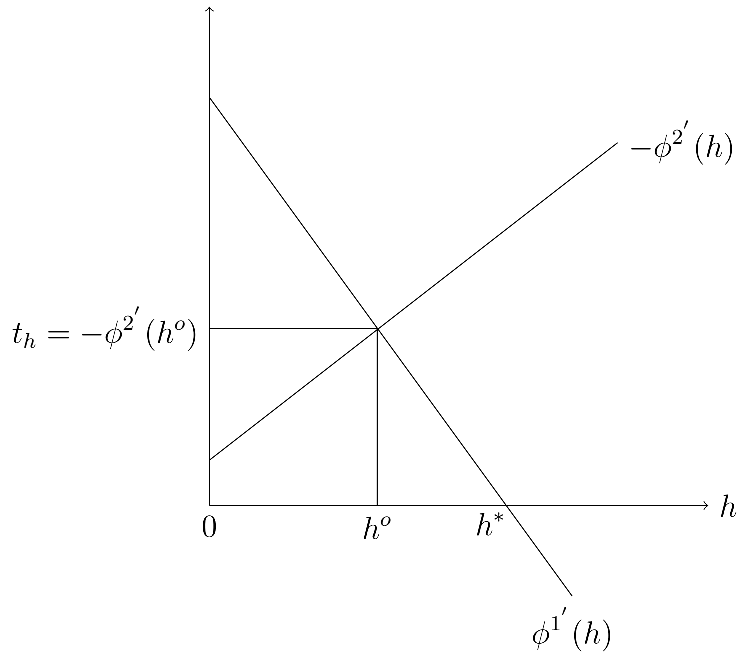


Figure 7.5: The Optimality-Restoring Pigouvian Tax

7.2.2.1 Quotas and Taxes

To fix ideas, suppose that h generates a negative external effect, so that $h^o < h^*$. The most direct sort of government intervention to achieve efficiency is to directly control the externality-generating activity itself. The government can simply mandate that h be no larger than h^o, its optimal level. With this constraint, consumer 1 will indeed fix the level of the externality at h^o.

A second option is for the government to attempt to restore optimality by imposing a (Pigouvian) tax on the externality-generating activity. As we have seen that tax rate $t_h = -\phi^{2'}(h^o) > 0$. Consumer 1's chooses the level of h to

$$\max_{h \geq 0} \phi^1(h) - t_h h,$$

Given $t_h = -\phi^{2'}(h^o) > 0$, consumer 1 chooses h^o as the optimum level of h (see figure 7.5). Note that the optimality-restoring tax is exactly equal to the marginal externality at the optimal solution. That is, it is exactly equal to the amount that consumer 2 would be willing to pay to reduce h slightly from its optimal level h^o. When faced with this tax. consumer 1 is effectively led to carry out an individual cost benefit computation that *internalizes* the externality that she imposes on consumer 2.

The principles for the case of a positive externality are exactly the same, only now when we set $t_h = -\phi^{2'}(h^o) < 0$, t_h takes the form of a per-unit *subsidy* (i.e. consumer 1 *receives* a payment for each unit of the externality she generates).

A point to note here is that, in general, it is essential to tax the externality-producing activity directly. For instance, suppose that, in the polluting firm example. A tax on its output leads the firm to reduce its output level but may not have any effect (or, more generally, may have too little effect) on its pollution emissions. Taxing output achieves optimality only in the special case in which emissions bear a fixed monotonic relationship to the level of output. In this special case, emissions can be measured by the level of output, and a tax on output is essentially equivalent to a tax on emissions.

7.2.3 Resolution of Consumption Externalities: Property Rights and the Coase Theory

Suppose that consumer 2 has enforceable property rights with regard to the externality-generating activity. Then consumer 1 is unable to engage in the externality-producing activity without consumer 2's permission. Suppose, they engage in bargaining and consumer 2 makes consumer 1 a take-it-or-leave-it offer, demanding a payment of T in return for permission to generate externality level h. Consumer 1 will agree to this demand if and only if she will be at least as well off as she would be by rejecting it, that is, if and only if $\phi^1(h) - T \geq \phi^1(0)$. Hence, consumer 2 will choose her offer (h, T) to solve

$$\max_{T, h \geq 0} \phi^2(h) + T$$
$$\text{subject to } \; \phi^1(h) - T \geq \phi^1(0).$$

Since the constraint is binding in any solution to this problem $T = \phi^1(h) - \phi^1(0)$. Therefore, consumer 2's optimal offer involves the level of h that solves

$$\max_{h \geq 0} \phi^2(h) + \phi^1(h) - \phi^1(0).$$

But this is precisely h^o, the socially optimal level.

Moreover, as we have observed that the precise allocation of the rights between the two consumers is inessential to the achievement of optimality. Suppose, for example, that consumer 1 instead has the right to generate as much of the externality as she wants. In this case, in the absence of any agreement, consumer 1 will generate externality level h^*. Now consumer 2 will need to offer a $T < 0$ (i.e., to pay consumer 1) to have it $h < h^*$. In particular, consumer 1 will agree to externality level h if and only if $\phi^1(h) - T \geq \phi^1(h^*)$. As a consequence, consumer 2 will offer to set h at the level that solves

$$\max_{h \geq 0} \phi^2(h) + \phi^1(h) - \phi^1(h^*).$$

Once again, the optimal extérnality level h^o results. *The allocation of rights affects only the final wealth of the two consumers* by altering the payment made by consumer 1 to consumer 2. In the first case, consumer 1 pays $\phi^1(h^o) - \phi^1(0) > 0$ to be allowed to set $h^o > 0$, whereas in the second, she "pays" $\phi^1(h^o) - \phi^1(h^*) < 0$ in return for setting

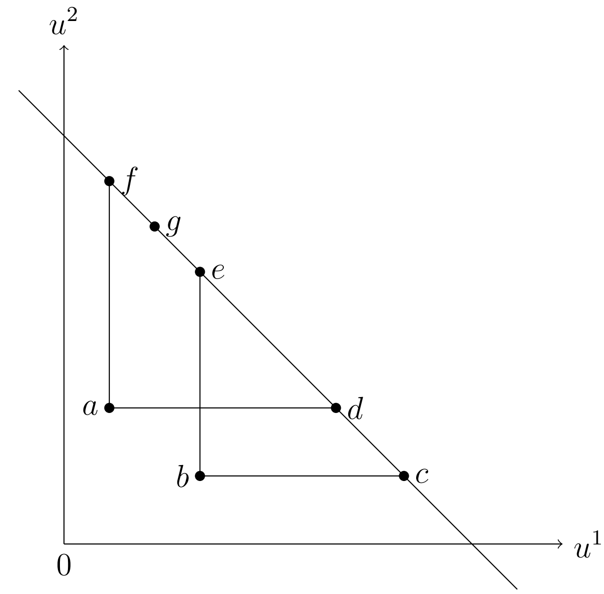


Figure 7.6: The Final Distribution of Utilities Under Different Property Rights Institutions and Different Bargaining Procedure

$h^o < h^*$.

All this is illustrated in Figure 7.6, in which we represent the utility possibility set for the two consumers. Every point in the boundary of this set corresponds to an allocation with externulity level it h^o. The points a and b correspond to the utility levels arising, respectively, from externality levels 0 and h^* in the absence of any transfers. They constitute the initial situation after the assignment of property rights (to consumers 2 and 1, respectively) but before bargaining. In the particular bargaining procedure we have adopted (which gives the power to make a take-it-or-leave-it offer to consumer 2), the utility levels after bargaining are points f and e, respectively. if the bargaining power (i.e., the power to make the take-it-or-leave-it offer) had been instead in the hands of consumer l, the post-bargaining utility levels would have been points d and c, respectively.

Note that all three approaches require that the externality-generating activity be measureable. This is not a trivial requirement; in many cases, such measurement may be either technologically infeasible or very costly (consider the cost of measuring air pollution or noise). A proper computation of costs and benefits should take these costs into account. If measurement is very costly, then it may be optimal to simply allow the externality to persist.

7.2.4 Externalities and Missing Markets

The observation that bargaining can generate an optimal outcome suggests a connection between externalities and missing markets. After all, a market system can be viewed as a particular type of trading procedure.

Suppose that *property rights are well defined and enforceable* and that a competitive market for the right to engage in the externality-generating activity exists. For simplicity, we assume that consumer 2 has the right to an externality-free environment. Let p_h denote the price of the right to engage in one unit of the activity. In choosing how many of these rights to purchase, say h_1, consumer 1 will solve

$$\max_{h_1 \geq 0} \phi^1(h_1) - p_h h_1$$

The F.O.C. is $\phi^{1'}(h_1) = p_h$.

In deciding how many rights to sell, h_2 say, consumer 2 will solve

$$\max_{h_2 \geq 0} \phi^2(h_2) + p_h h_2$$

The F.O.C. is $\phi^{2'}(h_2) = -p_h$.

In a competitive equilibrium, the market for these rights must clear; that is, we must have $h_1 = h_2$ which implies that the level of rights traded in this competitive rights market, say h^{**}, satisfies

$$\phi^{1'}(h^{**}) = -\phi^{2'}(h^{**})$$

Observe, that this is the same condition we had earlier, hence $h^{**} = h^o$. The equilibrium price of the externality is $p_h^* \phi^{1'}(h^o) = -\phi^{2'}(h^o)$.

Consumer 1 and 2's equilibrium utilities are-then $\phi^1(h^o) - p_h^* h$ and $\phi^2(h^o) + p_h^* h$, respectively. The market therefore works as a particular bargaining procedure for splitting the gains from trade; for example, point g in Figure 7.6 could represent the utilities in the competitive equilibrium.

We see that if a competitive market *exists* for the externality, then optimality results. Thus, externalities can be seen as being inherently tied to the absence of certain competitive markets, a point originally noted by Meade (1952) and substantially extended by Arrow (1969). Indeed, recall that our original definition of an externality, explicitly required that an action chosen by one agent must directly affect the well-being or production capabilities of another. Once a market exists for an externality, however, each consumer decides for herself how much of the externality to consume at the going prices.

7.3 Consumption Externalities: Example II

To illustrate consumption externality let us imagine two roommates A and B who have preferences over "watching TV" (good 1) and "money" (good 2). We suppose that two types of programs are available on TV – while channel I telecasts only sports programs, the sole program of channel II is music. It is assumed that both consumers like money but that A likes to watch sports programs and dislikes music while B likes to listen to music. Let the utility functions of the two consumers be

$$u^A(x) = x_1^A x_2^A - x_1^B, \tag{7.3.1}$$

$$u^B(x) = x_1^B x_2^B. \tag{7.3.2}$$

where $x = (x_1^A, x_2^A; x_1^B, x_2^B)$ with x_1^i being the time in a day that roommate i ($i = \{A, B\}$) spends for watching TV and x_2^i is the amount of money possessed by roommate i. Since A dislikes music he gets irritated when B listens to music. This is expressed by showing dependence of A's utility on x_1^B, the time that B spends for listening to music, in a negative way. Thus, this is a case of negative consumption externality.

The initial endowments of the two consumers are $\omega^A = (10, 0)$ and $\omega^B = (0, 10)$, we mean that initially A does not possess any money though he owns the TV and the total time in a day for which TV programs are telecast i.e. 10 hours. In contrast $\omega^B = (0, 10)$ means that initially B owns the entire amount of money, which is 10. Trade in this economy will mean that A will allow B to watch TV for some time in exchange for money.

To start the analysis let us first solve for the set of Pareto efficient allocations, Since $x_1^A + x_1^B = 10$ and $x_2^A + x_2^B = 10$, we can rewrite $u^A(\cdot)$ as

$$u^A(x) = x_1^A x_2^A - (10 - x_1^A) == x_1^A x_2^A + x_1^A - 10.$$

Next we calculate the marginal rate of substitution of good 2 for good 1 for A:

$$\text{MRS for A} = \frac{\text{marginal utility for good1}}{\text{marginal utility for good2}} = \frac{\frac{\partial u^A(x)}{\partial x_1^A}}{\frac{\partial u^A(x)}{\partial x_2^A}} = \frac{x_2^A + 1}{x_1^A},$$

and for roommate B:

$$\text{MRS for B} = \frac{\text{marginal utility for good1}}{\text{marginal utility for good2}} = \frac{\frac{\partial u^B(x)}{\partial x_1^B}}{\frac{\partial u^A(x)}{\partial x_2^B}} = \frac{x_2^B}{x_1^B}.$$

Then, to determine the locus of tangency points of the two individuals' indifference curves, we get MRS for A equal to MRS for B:

$$\frac{x_2^A + 1}{x_1^A} = \frac{x_2^B}{x_1^B} = \frac{10 - x_2^A}{10 - x_1^A}.$$

Solving this in terms of x_1^A we get

$$x_2^A = (1 + \frac{1}{10})x_1^A - 1. \tag{7.3.3}$$

7.3.1 Nonoptimality of the Competitive Outcome

In competitive equilibrium persons A and B will act as price takers. Person A wants to maximize his utility

$$u^A(x) = x_1^A x_2^A - x_1^B \quad \text{subject to } p_1 x_1^A + p_2 x_2^A = 10p_1$$

, where p_i are the competitive price of good i.

Now, A's utility maximizing conditions are

$$\frac{\text{marginal utility for good1}}{\text{marginal utility for good2}} = \frac{p_1}{p_2} \text{ and } p_1 x_1^A + p_2 x_2^A = 10p_1.$$

On substituting the expressions for marginal utilities of goods 1 and 2 for A, we can rewrite the above conditions as

$$\frac{x_2^A}{x_1^A} = \frac{p_1}{p_2} \text{ and } p_1 x_1^A + p_2 x_2^A = 10p_1.$$

Person B, on the other hand, wants to maximize his utility

$$u^B(x) = x_1^B x_2^B \quad \text{subject to } p_1 x_1^B + p_2 x_2^B = 10p_2.$$

Since the marginal utilities of good 1 and 2 for B are x_2^B and x_1^B respectively, the utility maximizing conditions for this person are

$$\frac{x_2^B}{x_1^B} = \frac{p_1}{p_2} \text{ and } p_1 x_1^B + p_2 x_2^B = 10p_2.$$

The unique competitive equilibrium allocation, thus, is $((x_1^A = 5, x_2^A = 5), (x_1^B = 5, x_2^B = 5))$ and the competitive price vector is $p_1 = p_2 = 1$. Clearly, the unique competitive equilibrium allocation $((5,5),(5,5))$ does not satisfy the tangency condition.

$$x_2^A = (1 + \frac{1}{10})x_1^A - 1$$

showing when externalities are present, a competitive equilibrium allocation need not be Pareto efficient, and the First Fundamental Theorem breaks down.

7.3.2 Resolution of Consumption Externalities: Pigouvian Taxes

Externalities weaken the link between competition and Pareto efficiency. There is a remedy that is consistent with the decentralized price mechanism. This remedy is the introduction of per unit (or marginal) taxes or subsidies on the consumption of goods that induce externalities. They will depend on the quantities consumed. Furthermore, they should be designed to encourage a person's consumption of a good if that consumption has positive external effects, and to discourage a person's consumption of a good if that consumption generates negative external effects.

Let us now return to the example of the previous section. In that example B's consumption of good 1 has a negative external effect on person A. But when person B makes his consumption decision in the standard competitive equilibrium framework, he does not take into account the negative impact of this consumption on A's utility. This suggests that this consumption good 1 should be taxed.

Let t be a tax paid by Person B for each unit Of good 1 that he consumes. The connection between competition and efficiency will be, reestablished if there exists a t, a price vector $p = (p_1, p_2)$ and cash transfers T_1, T_2 such that when A's budget constraint is $p_1x_1^A + p_2x_2^A \leq 10p_1 + T_1$ and B's budget constraint is $p_1x_1^B + p_2x_2^B \leq 10p_2 + T_2$ and they move to a Pareto efficient allocation through the competitive mechanism.

Next we now show that with appropriate choice of T_1 and T_2 we can move to any desired Pareto efficient allocation.

To see this, let $y^A = (5, 4.5)$ and $y^B = (5, 5.5)$. Since the allocation $y = (y^A, y^B)$ satisfies equation (7.3.3), it is Pareto efficient. Now, the marginal rate of substitution for person A at y^A is

$$\frac{4.5}{5} = \frac{p_1}{p_2}$$

and his budget constraint is

$$5p_1 + 4.5p_2 = 10p_1 + T_1.$$

If we let $T_1 = 0$, we have $\frac{p_1}{p_2} = \frac{4.5}{5}$. Thus, if we normalize prices by setting $p_2 = 10$, we have $p_1 = 9$.

Next, the marginal rate of substitution for person B at y^B is

$$\frac{5.5}{5} = \frac{p_1 + t}{p_2}$$

and his budget constraint is

$$5(p_1 + t) + 5.5p_2 = 10p_1 + T_1.$$

Given, $p_1 = 9$ and $p_2 = 10$, we must have $t = 2$ and $T_2 = 10$. Note that the total government cash transfer equals its commodity tax revenue.

Therefore, if $T_1 = 0$, $T_2 = 10$ and $t = 2$, the competitive mechanism, modified by T_1, T_2 and t, will take the economy to the Pareto efficient allocation (y^A, y^B). Thus, the introduction of the tax t reestablishes the connection between competition and efficiency.

To explain the intuitive reasoning behind this calculation, note that at y^A the marginal utility to person A of his own consumption of good 1 is 4.5. Now if person B increases his consumption of good 1 by 1 unit, person A would have to increase his consumption of good 1 by $\frac{1}{4.5}$ units in order to remain as well-off as before. Thus, there is a marginal rate of substitution between A's consumption of good 1 and B's consumption of good 1 and this is given by the marginal disutility to person A of person B's consumption of good 1 divided by the marginal utility to person A of his own consumption of good 1. This is a measure of the damage done to A by B's consumption of good 1. We call this marginal external damage or cost of person B's consumption of good 1. Note that this marginal external cost is measured in terms of units of good 1. Therefore, by multiplying this by the price of good 1, we get the marginal external cost in money terms. Since $p_1 = 9$, this amount turns out to be 2. But the required tax was 2. That is, the tax equals the value of the external damage or cost. This is intuitively reasonable: If person B is causing ₹2 worth of external damage for every extra unit of good 1 he consumes then the appropriate way to achieve efficiency is to impose a tax of ₹2 per unit on each extra unit he consumes. This way externality is internalized.

7.3.3 Resolution of Consumption Externalities: Property Rights and the Coase Theory

We can also apply the Coase theorem to internalize externalities here. Suppose person A has a legal right to silence. It can easily happen that A would prefer to trade some of his right to silence in exchange for some money. That is, A is allowing himself to be bribed to consume some of B's noise. It is easy to imagine that A and B are trading to a Pareto efficient point

Conversely, suppose B has a legal right to listen to loud music and (hence to noise) and A would have to bribe B to reduce noise. We can easily imagine the agents trading to a mutually preferred point.

The exact position of the efficient point is determined by the property rights and the precise mechanism which A and B use to trade. This is like the usual Edgeworth box analysis, but described in a somewhat different set-up. Thus, we see that whenever we have a market for noise, a competitive equilibrium becomes Pareto efficient.

Relevant Parts of the Reference Book (Serrano Feldman): Chapter 17.

Chapter 8

Public Goods

In the previous chapter we analyzed certain kinds of externalities, and discussed the mechanisms for correcting externality problems. However, our discussion has been concerned solely with private goods. (A good is said to be private if it conveys benefits only to the purchaser. For instance, a domestic car is a private good.) We will now examine the theory of production and consumption of goods whose character is public, rather than private.

How do we define a public good? Two important properties that characterize public goods are non-exclusivity and non-rivalry. A good is exclusive if it is easy to exclude individuals from benefitting from the good. For most private goods such exclusion is possible: I can easily be excluded from consuming a glass of milk if I do not pay for it. A good is non-exclusive if it is impossible to exclude individuals from benefiting from the good.

Definition 8.1. *A good is exclusive if it is relatively easy to exclude individuals from benefiting from the good once it is produced. A good is* **nonexclusive** *if it is impossible (or costly) to exclude individuals from benefiting from the good.*

A good is non-rival if one person's consumption does not decrease the amount available to others. Equivalently, we can say that a non-rival good is one for which additional units can be consumed at zero marginal social cost. A good is rival if one person's consumption reduces the amounts available to others. Consumption of additional units of rival goods involves some marginal costs of production. Consumption of one more glass of milk by someone, for example, requires that some resources be devoted to its production. Private goods are both exclusive and rival.

Definition 8.2. *A good is nonrival if consumption of additional units of the good involves zero social marginal costs of production.*

Many goods that are non-exclusive are also non-rival. National defence and mosquito control are two examples. Once a defence system is established, everyone in a country benefits from it. Similar comments apply to goods such as mosquito control or inoculation against diseases. In these cases, once the programs are implemented, nobody in the community can be excluded from those benefits. Similarly consumption of these goods by one person does not reduce the amount available to others. Thus, additional consumption of these goods takes place at zero marginal cost. Goods that are non-exclusive and non-rival are called public goods. Other examples of public goods are TV and radio broadcasts, highways, lighthouses, police and fire protection and so on.

Some goods may possess one property, but not the other. For instance, the use of a bridge during off-peak hours may be non-rival, but it is possible to exclude potential users by imposing toll booths. Goods of this type are called club goods. A cable TV broadcast is another example of the type of good. We may also have privately produced club goods such as I swimming pool or a golf course. In this case a private producer makes available a good which is consumed jointly by members.

We may have some goods that are not excludable, but are rival. That is, given the quantity of the good, allowing an additional individual to use the good does not prevent the previous users from using it, but it does reduce benefits attained by previous users. Therefore there is some opportunity cost involved in admitting additional users. This phenomenon of partial rivalness is called congestion in economics literature. The opportunity cost of allowing more users of the public good, or the reduction in benefits to those already consuming it, is called congestion cost. An example of a public good of this type is a crowded street, Any body can use it, but one person's use reduces the amount of space available to others. Sometimes goods of this type and club goods are referred to as impure public goods. On the other hand, non rival, non-exclusive goods are called pure public goods.

There exist public goods on the production side also. In this case a factor of production is used simultaneously by a number of firms. A good of this type is called public intermediate good. The fruits of new discoveries, public information and weather reports might be examples.

		Exclusive: Yes	Exclusive: No
Rival	Yes	Hot dogs, automobiles, houses	Fishing grounds, public grazing land, clean air
Rival	No	Bridges, swimming pools, satellite television transmission (scrambled)	National defense, mosquito control, justice

Figure 8.1: Examples Showing the Typology of Public and Private Goods

In this chapter, we study public goods in the following sense of the term.

Definition 8.3. *A good is a* **(pure) public good** *if, once produced, no one can be* *excluded from benefiting from its availability and if the good is nonrival – the marginal cost of an additional consumer is zero.*

Note that a public "good" need not necessarily be desirable; that is, we may have public bads (e.g., foul air).

8.1 Example I

Consider a setting with N consumers and one public good, in addition to L traded goods of the usual, private, kind. We again adopt a partial equilibrium perspective by assuming that the quantity of the public good has no effect on the prices of the L traded goods and that each consumer's utility function is quasilinear with respect to the same numeraire, traded commodity. As in Chapter 7, we can therefore define, for each consumer i, a derived utility function over the level of the public good. Letting x denote the quantity of the public good, we denote consumer i's utility from the public good by $\phi^i(x)$. We assume that this function is twice differentiable, with $\phi^{i''}(x) < 0$ at all $x \geq 0$. *Note that precisely because we are dealing with a public good, the argument x does not have an i subscript.*

The cost of supplying q units of the public good is $c(q)$. We assume that $c(\cdot)$ is twice differentiable, with $c''(q) > 0$ at all $q \geq 0$.

To describe the case of a desirable public good whose production is costly, we take $\phi^{i'}(\cdot) > 0$ for all i and $c'(\cdot) > 0$. The analysis, in general, applies equally well to the case of a public bad whose reduction is costly, where $\phi^{i'}(\cdot) < 0$ for all i and $c'(\cdot) < 0$.

8.1.1 Conditions For Pareto Optimality

In this quasilinear model, any Pareto optimal allocation maximizes aggregate surplus and therefore must involve a level of the public good that solves

$$\max_{q \geq 0} \sum_{i=1}^{N} \phi^i(q) - c(q).$$

The necessary and sufficient first-order condition for the optimal quantity q_o is then

$$\sum_{i=1}^{N} \phi^{i'}(q_o) \leq c'(q_o) \quad \text{with equality if } q_o > 0.^{1} \tag{8.1.1}$$

Condition (8.1.1) is the classic optimality condition for a public good first derived by Samuelson (1954; 1955). At an interior solution, we have $\sum_{i=1}^{N} \phi^{i'}(q_o) = c'(q_o)$, so that

[1]Whenever there is interior solution, that is $q_o > 0$ this condition holds with equality. We, in general, have solved all the problems with equality. But the general condition is this, that is when marginal benefit is smaller than marginal cost at all $q \geq 0$, it is better to choose $q = 0$.

at the optimal level of the public good the sum of consumers' marginal benefits from the public good is set equal to its marginal cost.

8.1.2 Inefficiency of Private Provision of Public Goods

Consider the circumstance in which the public good is provided by means of private purchases by consumers. We imagine that a market exists for the public good and that each consumer i chooses how much of the public good to buy, denoted by $x^i \geq 0$, taking as given its market price p. The total amount of the public good purchased by consumers is then $x = \sum_{i=1}^{N} x^i$. Formally, we treat the supply side as consisting of a single profit-maximizing firm with cost function $c(\cdot)$ that chooses its production level taking the market price as given.

At a competitive equilibrium involving price p^*, each consumer i's purchase of the public good x^{i^*} must maximize her utility and so must solve

$$\max_{x^i \geq 0} \sum_{i=1}^{N} \phi^i(x^i + \sum_{j \neq i} x^{j^*}) - p^* x^i. \tag{8.1.2}$$

In determining her optimal purchases, consumer i takes as given the amount of the private good being purchased by each other consumer (as in the Nash equilibrium concept we have studied). Consumer i's purchases x^{i^*} must therefore satisfy the necessary and sufficient first-order condition

$$\phi^{i'}(x^{i^*} + \sum_{j \neq i} x^{j^*}) \leq p^*, \text{ with equality if } x^{i^*} > 0. \tag{8.1.3}$$

Letting $x^* = \sum_{i=1}^{N} x^{i^*}$ denote the equilibrium level of the public good, for each consumer i we must therefore have

$$\phi^{i'}(x^*) \leq p^*, \text{ with equality if } x^{i^*} > 0. \tag{8.1.4}$$

The firm's supply, on the other hand, must solve

$$\max_{q \geq 0} p^* q - c(q) \tag{8.1.5}$$

and therefore must satisfy the standard necessary and sufficient first-order condition

$$p^* \leq c'(q^*) \text{ with equality if } q^* > 0. \tag{8.1.6}$$

At a competitive equilibrium, $q^* = x^*$. Thus, letting $\delta^i = 1$ if $x^{i^*} > 0$ and $\delta^i = 0$ if

$x^{i^*} = 0$, (8.1.3) and (8.1.6) tell us that

$$\sum_{i=1}^{N} \delta^i [\phi^{i'}(q^*) - c'(q)] = 0.$$

Recalling that $\phi'(\cdot) > 0$ and $c'(\cdot) > 0$, this implies that whenever $N > 1$ and $q^* > 0$ (so that $\delta^i = 1$ for some i), we have

$$\sum_{i=1}^{N} \phi^{i'}(q^*) > c'(q^*) \tag{8.1.7}$$

Comparing (8.1.7) with (8.1.1), we see that whenever $q_o > 0$ and $N > 1$, the level of the public good provided is too low; that is $q^* < q_o$.[2]

The cause of this inefficiency can be understood in terms of our discussion of externalities in Chapter 7. Here each consumer's purchase of the public good provides a direct benefit not only to the consumer herself but also to every other consumer. Hence private provision creates a situation in which externalities are present. The failure of each consumer to consider the benefits for others' of her public good provision is often referred to as the *free-rider problem: Each consumer has an incentive to enjoy the benefits of the public good provided by others while providing it insufficiently herself.*

In fact in the present model the free-rider problem takes a very stark form. To see this most simply, suppose that we can order the consumers according to their marginal benefits, in the sense that $\phi^{1'} < .... < \phi^{N'}$ at all $x \geq 0$. Then condition (8.1.3) can hold with equality only for a single consumer and, moreover, this must be the consumer labeled N. Therefore, only the consumer who derives the largest (marginal) benefit from the public good will provide it; all others will set their purchases equal to zero in the equilibrium. The equilibrium level of the public good is then the level q^* that satisfies $\phi^{N'}(q^*) = c'(q^*)$. Figure 8.2 depicts both this equilibrium and the Pareto optimal level. Note that the curve representing $\sum_{i=1}^{N} \phi^{i'}(q)$ geometrically corresponds to a vertical *summation* of the individual curves representing $\phi^i(q)$ for $i = 1, ..., N$ (whereas in the case of a private good, the market demand curve is identified by adding the individual demand curves *horizontally*).

To be inserted.

Figure 8.2: Private Provision Leads to an Insufficient Level of a Desirable Public Good

[2] The conclusion follows immediately if $q_o = 0$. So suppose instead that $q^* > 0$. Then since, $\sum_{i=1}^{N} \phi^{i'}(q^*) - c'(q^*) > 0$ and $\sum_{i=1}^{N} \phi^{i'}(\cdot) - c'(\cdot)$ is decreasing, any solution to (8.1.1) must have a larger value than q^*. Note that, in contrast, if we are dealing with a public bad, so that $\phi^{i'}(\cdot) < 0$ and $c'(\cdot) < 0$, then the inequalities reverse and $q_o < q^*$.

The inefficiency of private provision is often remedied by governmental intervention in the provision of public goods. Just as with externalities, this can happen not only through quantity-based intervention (such as direct governmental provision) but also through "price-based" intervention in the form of taxes or subsidies. For example, suppose that there are two consumers with benefit functions $\phi^1(x_1 + x_2)$ and $\phi^2(x_1 + x_2)$, where x_i is the amount of the public good purchased by consumer i, and that $q_o > 0$. By analogy with the analysis in Chapter 7, a subsidy to each consumer i per unit purchased $s^i = \phi^{-i^{\prime}}(q_o)$ [or, equivalently, a tax of $\phi^{-i^{\prime}}(q_o)$ per unit that consumer i's purchases of the public good fall below some specified level] faces each consumer with the marginal external effect of her actions and so generates an optimal level of public good provision by consumer i. Formally, if $(\tilde{x}^1, \tilde{x}^2)$ are the competitive equilibrium levels of the public good purchased by the two consumers given these subsidies, and if $\tilde{p}$ is the equilibrium price, then consumer i's purchases of the public good, $\tilde{x}^i$, must solve

$$\max_{x^i \geq 0} \phi^i(x^i + \tilde{x}^j) + s^i x^i - \tilde{p} x^i,$$

and so $\tilde{x}^i$, must satisfy the necessary and sufficient first-order condition

$$\phi^{i^{\prime}}(\tilde{x}^1 + \tilde{x}^2) + s^i \leq \tilde{p}, \quad \text{with equality of } \tilde{x}^i > 0.$$

Substituting for s^i, and using both condition (8.1.6) and the market-clearing condition that $\tilde{x}^1 + \tilde{x}^2 = \tilde{q}$, we conclude that $\tilde{q}$ is the total amount of the public good in the competitive equilibrium given these subsidies if and only if

$$\phi^{i^{\prime}}(\tilde{q}) + \phi^{-i^{\prime}}(q_o) \leq c^{\prime}(\tilde{q}),$$

with equality for some i if $\tilde{q} > 0$. Recalling (8.1.1) we see that $\tilde{q} = q_o$.

Note that both optimal direct public provision and this subsidy scheme require that the government know the benefits derived by consumers from the public good.

8.1.3 Lindahl Equilibria

Although private provision of the sort studied above results in an inefficient level of the public good, there is *in principle* a market institution that can achieve optimality. Suppose that, for each consumer i, we have a market for the public good "as experienced by consumer i." That is, we think of each consumer's consumption of the public good as a distinct commodity with its own market. We denote the price of this personalized good by p^i. Note that p^i may differ across consumers. Suppose also that, given the equilibrium price $p^{i^{**}}$, each consumer i sees herself as deciding on the *total amount of the public good* *she will consume*, x^i, so as to solve

$$\max_{x^i \geq 0} \phi^i(x^i) - p^{i^{**}} x^i.$$

Her equilibrium consumption level $x^{i^{**}}$ must therefore satisfy the necessary and sufficient first-order condition

$$\phi^{i'}(x^{i^{**}}) \leq p^{i^{**}} \quad \text{with equality if } x^{i^{**}} > 0.$$

The firm is now viewed as producing a bundle of N goods with a fixed-proportions technology (i.e., the level of production of each personalized good is necessarily the same). Thus, the firm solves

$$\max_{q \geq 0} \sum_{i=1}^{N} p^{i^{**}} q - c(q).$$

The firm's equilibrium level of output q^{**} therefore satisfies the necessary and sufficient first-order condition

$$\sum_{i=1}^{N} p^{i^{**}} \leq c'(q^{**}), \quad \text{with equality if } q^{**} > 0. \tag{8.1.8}$$

Comparing (8.1.8) with (8.1.1), we see that the equilibrium level of the public good consumed by each consumer is exactly the efficient level: $q^{**} = q_o$.

This type of equilibrium in personalized markets for the public good is known as a Lindahl equilibrium, after Lindahl (1919). To understand why we obtain efficiency, note that once we have defined personalized markets for the public good, each consumer, taking the price in her personalized market as given, fully determines her own level of consumption of the public good; externalities are eliminated.

Yet, despite the attractive properties of Lindahl equilibria; their realism is questionable. Note, first, that the ability to exclude a consumer from use of the public good is essential if this equilibrium concept is to make sense; otherwise a consumer would have no reason to believe that in the absence of making any purchases of the public good she would get to consume none of it.[3] Moreover, even if exclusion is possible, these are markets with only a single agent on the demand side. As a result, price-taking behavior of the sort presumed is unlikely to occur.

8.2 Example II

There are two individuals (A and B) and two goods (y and x) in the economy. Good y is an ordinary private good, and each person begins with an allocation of this good

[3] Thus, the possibility of exclusion can be important for efficient supply of the public good, even though the use of an exclusion technology is itself inefficient (a Pareto optimal allocation cannot involve any exclusion).

given by y^A and y^B, respectively. Each person may choose to consume some of his or her y directly or to devote some portion of it to the production of the public good, x. The amounts contributed are given by y_s^A and y_s^B, and the public good is produced according to the production function

$$x = f(y_s^A + y_s^B).$$

Resulting utilities for these two people in this society are given by

$$u^A(x, y^A - y_s^A) \qquad \text{and} \quad u^B(x, y^B - y_s^B). \tag{8.2.1}$$

Notice here that the level of public good production, x, enters identically into each person's utility function. This is the way in which the nonexclusivity and nonrivalry characteristics of such goods are captured mathematically. Nonexclusivity is reflected by the fact that each person's consumption of x is the same and independent of what he or she contributes individually to its production. Nonrivalry is shown by the fact that the consumption of x by each person is identical to the total amount of x produced. Consumption of x benefits by A does not diminish what B can consume. These two characteristics of good x constitute the barriers to efficient production under most decentralized decision schemes, including competitive markets.

The necessary conditions for efficient resource allocation in this problem consist of choosing the levels of public goods subscriptions (y_s^A and y_s^B) that maximize, say, A's utility for any given level of B's utility. The Lagrangian expression for this problem is

$$\mathcal{L} = u^A(x, y^A - y_s^A) + \lambda[u^B(x, y^B - y_s^B) - k] \tag{8.2.2}$$

where k is a constant level of B's utility. The first-order conditions for a maximum are

$$\frac{\partial \mathcal{L}}{\partial y_s^A} = u_1^A f' - u_2^A + \lambda u_1^B f' = 0 \tag{8.2.3}$$
$$\frac{\partial \mathcal{L}}{\partial y_s^B} = u_1^A f' + \lambda u_1^B f' - \lambda u_2^B = 0 \tag{8.2.4}$$
$$\tag{8.2.5}$$

A comparison of these two equations yields the immediate result that

$$\lambda u_2^B = u_2^A$$

As might have been expected here, optimality requires that the marginal utility of y consumption for A and B be equal except for the constant of proportionality, λ. This equation may now be combined with either Equation (8.2.3) or (8.2.4) to derive the optimality condition for producing the public good x. Using Equation (8.2.3), for example, gives

$$\frac{u_1^A}{u_2^A} + \frac{u_1^B}{u_2^B} = \frac{1}{f'} \tag{8.2.6}$$

or, more simply,

$$MRS^A + MRS^B = \frac{1}{f'}. \tag{8.2.7}$$

8.2.1 Failure of A Competitive Market

Production of goods x and y in competitive markets will fail to achieve this allocational goal. With perfectly competitive prices p_x and p_y, each individual will equate his or her MRS to the price ratio $\frac{p_x}{p_y}$. A producer of good x would also set $\frac{1}{f'}$ to be equal to $\frac{p_x}{p_y}$, as would be required for profit maximization.

This behavior would not achieve the optimality condition expressed in Equation (8.2.7). The price ratio $\frac{p_x}{p_y}$ would be "too low" in that it would provide too little incentive to produce good x. In the private market, a consumer takes no account of how his or her spending on the public good benefits others, so that consumer will devote too few resources to such production.

The allocational failure in this situation can be ascribed to the way in which private markets sum individual demands. For any given quantity, the market demand curve reports the marginal valuation of a good. If one more unit were produced, it could then be consumed by someone who would value it at this market price. For public goods, the value of producing one more unit is in fact the sum of each consumer's valuation of that extra output, because all consumers will benefit from it. In this case, then, individual demand curves should be added vertically (as shown in Figure 8.3) rather then horizontally (as they are in competitive markets). The resulting price on such a public good demand curve will then reflect, for any level of output, how much an extra unit of output would be valued by all consumers. But the usual market demand curve will not properly reflect this full marginal valuation.

8.2.2 Inefficiency of a Nash equilibrium

An alternative approach to the production of public goods in competitive markets might rely on individuals' voluntary contributions. Unfortunately, this also will yield inefficient results. Consider the situation of person A, who is thinking about contributing s^A of his or her initial y^A endowment to public goods production. The utility maximization problem for A is then

$$\max_{s^A} u^A(f(s^A, s^B), y^A - s^A) \tag{8.2.8}$$

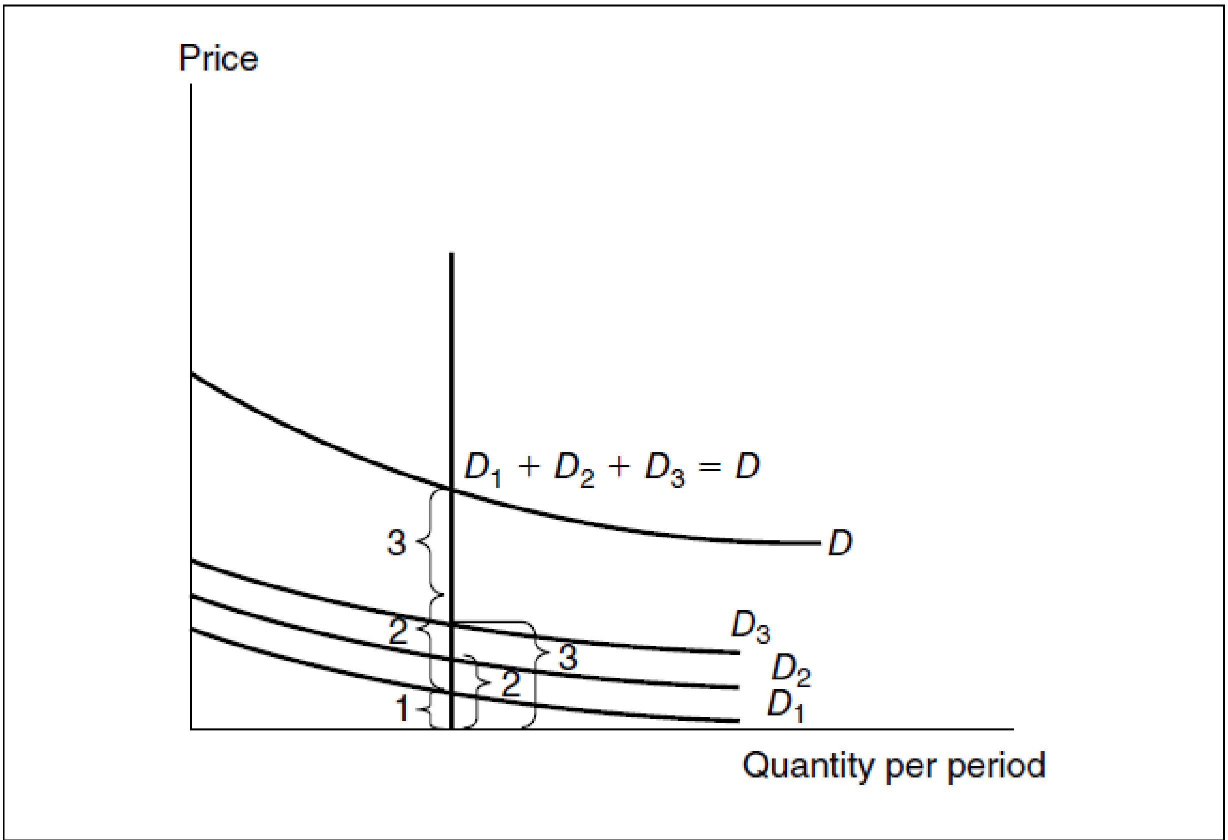


Figure 8.3: Derivation of the Demand for a Public Good

The first-order condition for a maximum is

$$u_1^A f' - u_2^A = 0 \quad \Rightarrow \quad \frac{u_1^A}{u_2^A} = MRS^A = \frac{1}{f'}. \tag{8.2.9}$$

Because a similar logic will apply to person B, efficiency condition (8.2.7) will once more fail to be satisfied. Again the problem is that each person considers only his or her benefit from investing in the public good, taking no account of the benefits provided to others. With many consumers, this direct benefit may be very small indeed. (For example, how much do one person's taxes contribute to national defense?) In this case, any one person may opt for $s^A = 0$ and become a pure *free rider*, hoping to benefit from the expenditures of others. If every person adopts this strategy, then no resources will be subscribed to public goods.

8.2.3 Lindahl Pricing of Public Goods

Each individual would be presented by the government with the proportion of a public good's cost he or she would be expected to pay and then reply (honestly) with the level of public good output he or she would prefer. In the notation of our simple general equilibrium model, individual A would be quoted a specific percentage α^A and then asked the level of public goods that he or she would want given the knowledge that this fraction of total cost would have to be paid. To answer that question (truthfully), this person would choose that overall level of public goods output, x to

$$\max_x u^A(x, y^A - \alpha^A f^{-1}(x)) \tag{8.2.10}$$

The first-order condition for this utility-maximizing choice of x is given by

$$u_1^A - \alpha^A u_2^B \cdot \frac{1}{f'} = 0 \quad \Rightarrow \quad MRS^A = \frac{\alpha^A}{f'} \tag{8.2.11}$$

Individual B, presented with a similar choice, would opt for a level of public goods satisfying

$$MRS^B = \frac{\alpha^B}{f'} \tag{8.2.12}$$

An equilibrium would then occur where $\alpha^A + \alpha^B = 1$ – that is, where the level of public goods expenditure favored by the two individuals precisely generates enough in tax contributions to pay for it. For in that case

$$MRS^A + MRS^B = \frac{\alpha^A + \alpha^B}{f'} = \frac{1}{f'} \tag{8.2.13}$$

and this equilibrium would be efficient (see Equation 8.2.7). Hence, at least on a conceptual level, the Lindahl approach solves the public good problem. Presenting each person with the equilibrium tax share "price" will lead him or her to opt for the efficient level of public goods production.

8.2.4 Shortcomings of the Lindahl solution

Unfortunately, Lindahl's solution is only a conceptual one. Because it is difficult to get true information – individuals know their tax shares will be based on their reported demands for public goods, they have a clear incentive to understate their true preferences – in so doing they hope that the "other guy" will pay. Hence, simply asking people about their demands for public goods should not be expected to reveal their true demands. In general, then, Lindahl's solution remains a tantalizing but not readily achievable target.

8.2.5 Local public goods[4]

Some economists believe that demand revelation for public goods may be more tractable at the local level. Because there are many communities in which individuals might reside, they can indicate their preferences for public goods (that is, for their willingness to pay Lindahl tax shares) by choosing where to live. If a particular tax burden is not utility

[4] Not in the syllabus

maximizing then people can, in principle, "vote with their feet" and move to a community that does provide optimality. Hence, with perfect information, zero costs of mobility, and enough communities, the Lindahl solution may be implemented at the local level. Similar arguments apply to other types of organizations (such as private clubs) that provide public goods to their members; given a sufficiently wide spectrum of club offerings, an efficient equilibrium might result. Of course, the assumptions that underlie the purported efficiency of such choices by individuals are quite strict. Even minor relaxation of these assumptions may yield inefficient results owing to the fragile nature of the way in which the demand for public goods is revealed.

8.2.6 Voting[5]

Since private provision of public good and the Lindahl tax scheme do not work very nicely, we must look elsewhere for an ideal public finance scheme. One mechanism can be voting. Voting is used as a social decision process many institutions.

Since voting is not very interesting for a two-person case, we will assume that there are N persons, where N is odd. The voting principle that we are going to discuss here as a social decision criterion yields a positive result when N is even also. A discussion on this is beyond the scope of this course. The interested readers may look at Black's book (Black, 1958). Suppose that these N persons are voting on the amount of a public good, more precisely, on the magnitude of expenditure on some public good.

Each person has preferences over alternative levels of expenditure. He has a most-preferred level of expenditure and his valuation about other expenditure levels will depend on their closeness to the most preferred level. Decisions will be made by majority rule, that is, we will assume that policies will be adopted if they are favored by the next whole integer above $\frac{N}{2}$ voters or more. (Thus, if there are $N = 5$ persons, then a policy will be adopted if it is voted for by 3 persons or more.)

M. de Condorcet, a French social theorist, observed an important peculiarity of the majority voting systems in the 1780s. The Condorcet observation says that a majority voting may not yield an equilibrium but instead may cycle among alternative options. To illustrate the Condorcet paradox, which is popularly referred to as the paradox of voting, suppose that there are three voters (A, B and C) choosing among three policy options representing three levels of spending on a particular public good (low (l), medium (m), or high (h)). Preferences of A, B and C among the three policy options are as shown in Figure 8.4. Now consider a vote between policy options l and m. Here l would win since it is favored by A and C and opposed by only B. In a vote between m and h, m would win, again by 2 votes to 1. But if we consider a vote between l and h, then h would win and consequently social choices would cycle. (The social preference relation does not satisfy the transitivity condition.)

[5] Not in the syllabus

Preferences	A	B	C
	l	m	h
	m	h	l
	h	l	m

Figure 8.4: Paradox Of Voting

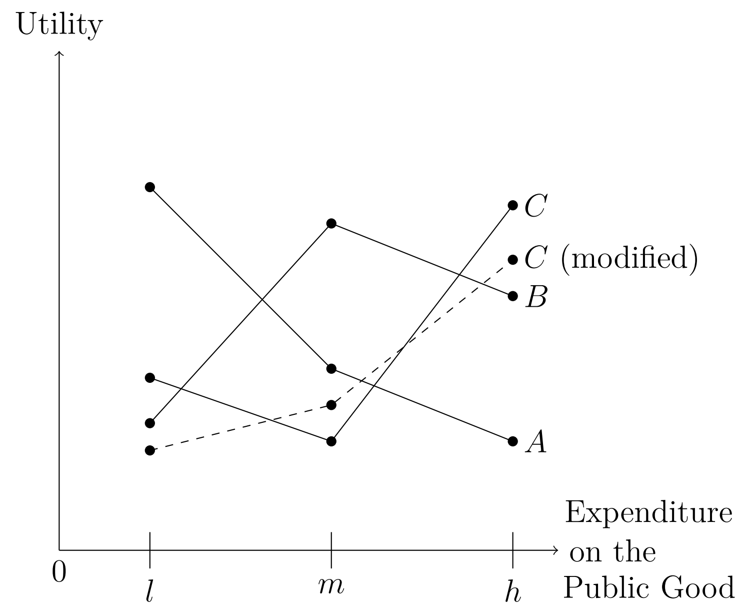


Figure 8.5: Single-Peaked Preference and Median Voter

The paradox of voting is quite disturbing. One may therefore ask what restrictions on preferences will rule it out. A fundamental result established by Black (1958) showed that equilibrium voting outcomes can always occur in cases where voters' preferences are "single-peaked". To understand single peakedness, we first illustrate in Figure 8.5 the preferences of the three voters considered above by assigning hypothetical utility levels to options l, m and h. The utility levels are assumed to be consistent with the preferences recorded in Table 8.4. For person A, utility decreased consistently as the expenditure on the public good increased.

To explain this, note that each person cares about the level of the public good and also about the amount he has to contribute to it. A higher level of expenditure means a higher amount of the public good but individuals also have to pay higher taxes to finance the good. Since A's utility decreases uniformly over expenditure levels, it seems he is more concerned about expenditures on the public good than the benefits he can derive from it. For person B, the utility of expenditure on the public good rises at first due to benefits of the public good but then it falls gradually due to the cost of providing it. Clearly, person C is more concerned about the benefits than the costs.

For persons A and B preferences are single-peaked. With single-peaked preferences, the utility of different expenditure levels should be either monotone (uniformly decreasing or increasing) or it should rise until the most preferred point and then fall. Single-peakedness thus requires that utility should not go up, down and then up again. Person C's preferences fail to satisfy this property and hence they are not single-peaked.

If each individual has single-peaked preferences, than social preferences revealed by majority voting will not exhibit intransitivity (Black, 1958). To see this suppose that C had preferences represented by the broken line in Figure 8.5. In that case, option m would win since it would defeat both l and h by 2 votes to 1. That is, the choice of the 'median' voter (B) whose preferences are between the preferences of A and the (modified) preferences of C wins. The choice of the median expenditure means that (excluding the median person) one-half of the population is willing to spend more and one-half wants to spend less. This result is intuitively reasonable. In a vote between the median expenditure and any amount smaller than that, the former wins, since the median person and persons richer than him will vote in its favor. Similarly, if there is a vote between the median expenditure and any amount higher than that, then the median person and everyone poorer than him will favor the median expenditure and hence it wins.

There are certain advantages of the median expenditure system of public finance. First, it is relatively simple. It can be easily understood by the people reporting their desired expenditures on the public good and voting on these expenditures.

But there are certain disadvantages also. The median outcome means that $\frac{N-M}{2}$ persons want more and $\frac{N-M}{2}$ persons want less, where M is the median person. It does not state explicitly how much more of the public good is wanted. Since the Samuelson optimality condition takes this kind of information into account, voting may not lead to an efficient level of the public good. Furthermore, people may have incentives to misrepresent their true preferences. As a consequence, the final outcome may be manipulated.

8.3 Example III

Consider a situation in which N citizens/agents/players must decide whether to undertake a public project, such as building a bridge, whose cost must be funded by the agents themselves. An outcome is a vector $x = (k, t^1, ..., t^N)$, where $k \in \{0, 1\}$ is the decision whether to build the bridge ($k = 1$ if the bridge is built and $k = 0$ if not) and $t^i \in \mathbb{R}$ is a monetary transfer to (or from, if $t^i < 0$) agent i. The cost of the project is $c > 0$ and so the set of feasible alternatives for the N agents is

$$X = \{(k, t^1, ..., t^N) : k \in \{0, 1\}, t^i \in \mathbb{R} \text{ for all } i, \text{ and } \sum_i t^i \leq -ck\}.$$

The constraint $\sum_i t_i \leq -ck$ reflects the fact that there is no source of outside funding for the agents (so that we must have $\sum_i t_i + k \leq 0$ if $k = 1$, and $\sum_i t_i \leq 0$ if $k = 0$). We assume that type θ^i's utility function has the quasilinear form

$$u^i(x, \theta^i) = \theta^i k + (w^i + t^i),$$

where w^i is agent i's initial endowment of numeraire ("money") and $\theta^i \in \mathbb{R}$. We can then interpret θ^i as agent i's willingness to pay for the bridge. The set of possible types for agent i is denoted Θ^i.

The *allocation rule* $k(\theta)$ is ex post efficient[6] if for all θ,

$$k(\theta) = \begin{cases} 1 & \text{if } \sum_i \theta^i \geq c, \\ 0 & \text{otherwise.} \end{cases} \tag{8.3.1}$$

$$\sum_i t^i(\theta) = -ck(\theta). \tag{8.3.2}$$

Suppose that the agents wish to implement allocation rule that satisfies (8.3.1) and (8.3.1) and in which an egalitarian contribution rule is followed, that is, in which $t^i(\theta) = -\frac{c}{N} k(\theta)$.

To consider a simple example, suppose that $\Theta^i = \{\bar{\theta}^i\}$ for $i \neq 1$ (so that all agents other than agent 1 have preferences that are known) and $\theta^1 \in [0, \infty)$. Suppose also that $c > \sum_{i \neq 1} \bar{\theta}^i > c\frac{N-1}{N}$. These inequalities imply, first, that with this allocation rule agent 1's type is critical for whether the bridge is built (if $\theta^1 \geq c - \sum_{i \neq 1} \bar{\theta}^i$ it is; if $\theta^1 < c - \sum_{i \neq 1} \bar{\theta}^i$ it is not), and that the sum of the utilities of agents $2, ..., N$ is strictly greater if the bridge is built under this egalitarian contribution rule than if it is not built (since $\sum_{i \neq 1} \bar{\theta}^i - c\frac{N-1}{N} > 0$).

Let us examine agent 1's incentives for truthfully revealing his type when $\theta^1 = c - \sum_{i \neq 1} \bar{\theta}^i + \epsilon$ for $\epsilon > 0$. If agent 1 reveals his true preferences, the bridge will be built because

$$[c - \sum_{i \neq 1} \bar{\theta}^i + \epsilon] + \sum_{i \neq 1} \bar{\theta}^i > c.$$

Agent l's utility in this case is

$$\begin{aligned} \theta^1 + w^1 - \frac{c}{N} &= [c - \sum_{i \neq 1} \bar{\theta}^i + \epsilon] + w^1 - \frac{c}{N} \\ &= [\frac{c(N-1)}{N} - \sum_{i \neq 1} \bar{\theta}^i + \epsilon] + w^i. \end{aligned}$$

[6] There are several stages at which economists typically think about Pareto efficiency, the *ex ante* stage prior to individuals finding out their types, the *interim* stage, where each knows only his own type, and the *ex post* stage when all types are known by all individuals.

But for $\epsilon > 0$ small enough, this is less than w^1, which is agent 1's utility if he instead claims that $\theta^1 = 0$, a claim that results in the bridge not being built. Thus, agent 1 will prefer not to tell the truth. Intuitively, under this allocation rule, when agent 1 causes the bridge to he built he has a positive externality on the other agents (in the aggregate). Because he fails to internalize this effect, he has an incentive to understate his benefit from the project.

8.3.1 The Vickrey-Clarke-Groves (VCG) Mechanism

The objective (of a social planner/ mechanism designer) is to design a *mechanism* (decision rule and transfer rule) so that the players truthfully reveal their types and *ex post efficient decision is implemented.*

The social planner asks the players to report their types and announces the following decision rule and transfer rule, given that the $(b^1, ..., b^N)$ are the reported types (observe, the reported type can be different from the true type).

$$\text{Decision Rule: } d = 1 \qquad \text{if and only if } \sum_i b^i \geq c$$

$$\text{Transfer Rule: } t^i = \begin{cases} \sum_{j \neq i} b^j - c & \text{if } \sum_i b^i \geq c \text{ (that is if } k = 1) \\ 0 & \text{otherwise.} \end{cases}$$

We now show that under this mechanism "truthful reporting" is weakly dominant.

Proposition 8.1. *In the VCG mechanism it is a weakly dominant strategy for each individual to report his type truthfully.*

Proof. The expected utility from truthful reporting is

$$EU^i(b^i = \theta^i) = \begin{cases} \theta^i + \sum_{j \neq i}^N b^j - c & \text{if } \theta^i + \sum_{j \neq i}^N b^j \geq c \;\; \Rightarrow \theta^i \geq c - \sum_{j \neq i}^N b^j \\ 0 & \text{if } \theta^i + \sum_{j \neq i}^N b^j < c \;\; \Rightarrow \theta^i < c - \sum_{j \neq i}^N b^j. \end{cases}$$

The expected utility from under reporting is

$$EU^i(b^i = b^{i'} < \theta^i) = \begin{cases} \theta^i + \sum_{j \neq i}^N b^j - c & \text{if } b^{i'} + \sum_{j \neq i}^N b^j \geq c \;\; \Rightarrow b^{i'} \geq c - \sum_{j \neq i}^N b^j \\ 0 & \text{if } b^{i'} + \sum_{j \neq i}^N b^j < c \;\; \Rightarrow b^{i'} < c - \sum_{j \neq i}^N b^j. \end{cases}$$

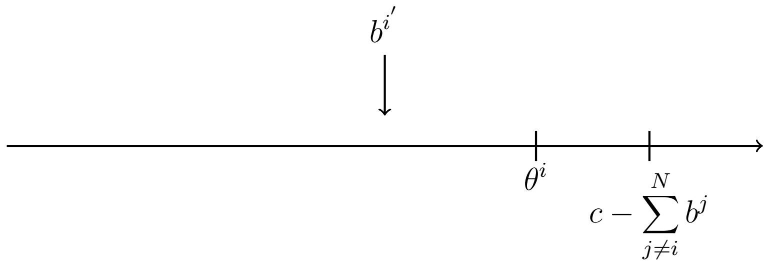
The expected utility from over reporting is

$$EU^i(b^i = b^{i''} < \theta^i) = \begin{cases} \theta^i + \sum_{j \neq i}^{N} b^j - c & \text{if } b^{i''} + \sum_{j \neq i}^{N} b^j \geq c \quad \Rightarrow b^{i''} \geq c - \sum_{j \neq i}^{N} b^j \\ 0 & \text{if } b^{i''} + \sum_{j \neq i}^{N} b^j < c \quad \Rightarrow b^{i''} < c - \sum_{j \neq i}^{N} b^j. \end{cases}$$

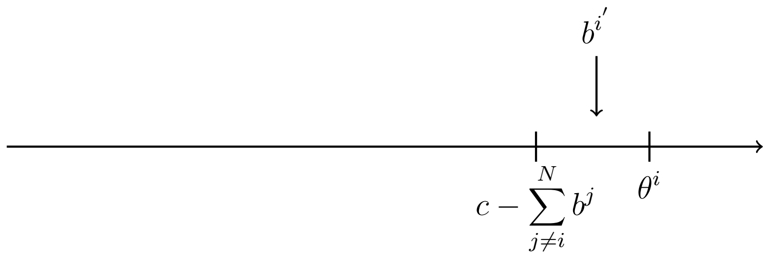
First we show that truthful reporting weakly dominates under reporting. Then, we show that truthful reporting weakly dominates over reporting.

Step 1. Truthful reporting weakly dominates under reporting

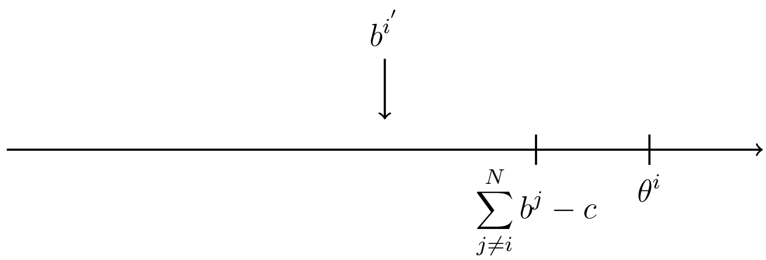
Case 1. $b^{i'} < \theta^i \leq c - \sum_{j \neq i}^{N} b^j$: $EU^i(b^i = \theta^i) = EU^i(b^i = b^{i'} < \theta^i) = 0$.



Case 2. $\theta^i > b^{i'} \geq c - \sum_{j \neq i}^{N} b^j$: $EU^i(b^i = \theta^i) = EU^i(b^i = b^{i'} < \theta^i) = \theta^i + \sum_{j \neq i}^{N} b^j - c$.



Case 3. $\theta^i > c - \sum_{j \neq i}^{N} b^j > b^{i'}$: $EU^i(b^i = \theta^i) = \theta^i + \sum_{j \neq i}^{N} b^j - c > 0 = EU^i(b^i = b^{i'} < \theta^i)$.

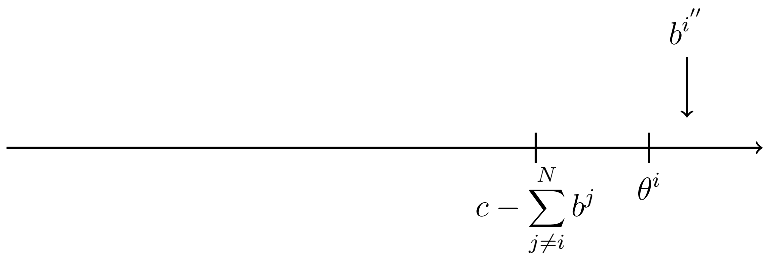


Thus, truthful reporting weakly dominates under reporting.

*Step 2.*Truthful reporting weakly dominates over reporting

Case 1. $\theta^i < b^{i''} \leq c - \sum_{j \neq i}^{N} b^j$: $EU^i(b^i = \theta^i) = EU^i(b^i = b^{i''} > \theta^i) = 0.$

Case 2. $b^{i''} > \theta^i \geq c - \sum_{j \neq i}^{N} b^j$: $EU^i(b^i = \theta^i) = EU^i(b^i = b^{i''} > \theta^i) = \theta^i + \sum_{j \neq i}^{N} b^j - c.$



Case 3. $b^{i''} > c - \sum_{j \neq i}^{N} b^j > \theta^i$: $EU^i(b^i = \theta^i) = 0 > \theta^i + \sum_{j \neq i}^{N} b^j - c = EU^i(b^i = b^{i''} > \theta^i).$

Thus, truthful reporting weakly dominates over reporting. ■

So, under this mechanism, $k(\theta) = 1$ whenever $\sum_i \theta^i \geq c$. However, there will be budget surplus in general, that is $\sum_i t^i \leq -ck(\theta)$.

8.3.1.1 Comments about VCG Mechanism

An interesting feature of the VCG mechanism is that it can be thought of as a generalisation of a second-price auction. Recall that in a second-price auction for a single good, the highest bidder wins and pays the second-highest bid. As we know, it is therefore a (weakly) dominant strategy for each bidder to bid his value, and so the bidder with highest value wins and pays the second highest value. This auction is sometimes described as one in which the winner *pays his externality.* The reason is that if the winner were not present, the bidder with second highest value would have won. Thus, the winning bidder, by virtue of his presence, precludes the second-highest value from being realised – he imposes an externality. Of course, he pays for the good precisely the amount of the externality he imposes, and the end result is efficient.

Now let us think about the externality imposed by each individual i on the remaining individuals. The trick to computing individual i's externality is to think about the difference his presence makes to the total utility of the others.
When individual i is present the total utility of the others is

$$\sum_{j \neq i} \theta^j k(\theta) - ck(\theta).$$

When individual i is not present the total utility of the others is

$$\sum_{j \neq i} \theta^j k(\theta^{-i}) - ck(\theta^{-i}).$$

Hence, the difference that i's presence makes to the total utility of the others is

$$\sum_{j \neq i} \theta^j k(\theta^{-i}) - ck(\theta^{-i}) - \Big[\sum_{j \neq i} \theta^j k(\theta) - ck(\theta)\Big].$$

We call this difference the *externality imposed by individual* i.

Note that one's externality is always non-negative and is typically positive because, by definition (of *ex post efficiency*)

$$k(\theta^{-i}) = \begin{cases} 1 & \text{if } \sum_{j \neq i} \theta^j \geq c, \\ 0 & \text{otherwise.} \end{cases}$$

So, when $k(\theta) = k(\theta^{-i})$, this externality is zero, whereas when $k(\theta) = 1$ and $k(\theta^{-i}) = 0$, then the player i's externality is

$$-\Big[\sum_{j \neq i} \theta^j - c\Big]$$

which is positive as $c > \sum_{j \neq i} \theta^j$ (that's why $k(\theta^{-i}) = 0$).

Relevant Parts of the Reference Book (Serrano Feldman): Chapter 18.

Chapter 9

Market Power

In this chapter we study the functioning of markets in which market power is present.

9.1 Monopoly

A single firm faces the entire market demand. Using its knowledge about this demand, the monopolist firm either chooses quantity or price in order to maximize its profit. Technically, a monopolist firm can choose that point on the market demand curve at which it prefers to operate. It may choose either price or equivalently quantity, but not both independently: It may choose the price and let the consumers decide how much they wish to buy at that price, or it may select the quantity. Whenever a firm can influence the price it receives for its product, the firm is said to have market power.

In our analysis, we will assume that the firm under consideration chooses quantity to maximize its profit (identical conclusions could equally well be developed where the monopolist chooses price to maximize its profit).

9.1.1 Costs and Revenues Under Monopoly

Let the inverse demand function be $p(q)$. Then, the monopolist's revenue is

$$R(q) = p(q) \cdot q.$$

Average revenue is defined as the revenue per unit of output.

$$AR(q) = \frac{R(q)}{q} = \frac{p(q) \cdot q}{q} = p(q).$$

Thus, the monopolist's average revenue curve is the market demand curve itself.

Marginal revenue at q, $MR(q)$ is the change in the revenue when the firm increases its output marginally.

$$MR(q) = \frac{\triangle R(q)}{\triangle q} = \frac{p(q) \triangle q + q \triangle p(q)}{\partial q} = p(q) + q\frac{\triangle p(q)}{\triangle q}$$

where $\frac{\triangle p(q)}{\triangle q}$ is the slope of the demand curve. Observe, if the monopolist firm decides to increase its output by $\triangle q$, there are two effects: First, the monopolist's revenue increases by the amount $p(q) \triangle q$ from that. Second, the monopolist pushes the price down by $\triangle p$ and gets lower price for all the units it was selling previously (that is q units) to sell the added $\triangle q$ units. Therefore, the total change in revenue $\triangle R(q)$ due to $\triangle q$ change in output is $\triangle R(q)$ due to $\triangle q$ change in output is $\triangle R(q) = p(q) \triangle q + q \triangle p$. Consequently, marginal revenue is $MR(q) = p(q) + q\frac{\triangle p(q)}{\triangle q}$.

Now compare this with the marginal revenue of a competitive seller: If a competitive firm lowers its price below the existing price charged by other firms, then it will capture the entire market from its competitors. But in the case of the monopoly, since the monopoly controls the whole market, it has to take care of the effect of a price reduction (quantity increase) on all units it was selling previously.

Elasticity and Revenue: Recap

Recall the definition of price elasticity of demand: The price elasticity of demand, e, is defined as the percentage change in price

$$e = \frac{\frac{\triangle q}{q}}{\frac{\triangle p}{p}}$$

If demand decreases significantly when the price increases, revenue will fall. On the other hand, if demand reduction is insignificant when price increases, revenue is expected to increase. Therefore, the direction of change in revenue is closely related to price elasticity.

Proposition 9.1. *An increase in the price of a good leads to an increase or a decrease in the total revenue, if the demand for the good is inelastic or elastic. If demand is of unitary elasticity, an increase in the price leads to no change in the revenue.*

Proof. Let R denote the revenue: $R = pq$. If we let the price change to $p + \triangle p$, where $\triangle p > 0$, and the quantity change to $q + \triangle q$, the new revenue is

$$R^{'} = (p + \triangle p)(q + \triangle q) = pq + q \triangle p + p \triangle q + \triangle p \triangle q$$

So, the change in revenue is

$$\triangle R = q \triangle p + p \triangle q + \triangle p \triangle q$$

For small values of $\triangle p$ and $\triangle q$, the last term on the right hand side of $\triangle R$ can be neglected. Hence

$$\triangle R \simeq q \triangle p + p \triangle q$$

This means that the change in total revenue is approximately equal the quantity times change in price plus price times change in quantity. Alternatively, we can write it as

$$\triangle R = q \triangle p[1 + \frac{p \triangle q}{q \triangle p}] = q \triangle p[1 + e]$$

Now by assumption $\triangle p > 0$. Therefore when demand is inelastic ($-1 < e \leq 0$), an increase ($\triangle p$) in p is accompanied by an increase ($\triangle R$) in revenue, R. If demand is elastic ($e < -1$), $\triangle p > 0$ implies that $\triangle R < 0$. When demand is of unitary elasticity ($e = 1$), $\triangle p > 0$ implies that $\triangle R = 0$. ■

The intuitive reasoning behind this result is quite clear. When demand is relatively unresponsive to price (inelastic demand), a price increase does not change the demand very much and total revenue rises. When demand is highly responsive to price (elastic demand), a price increase reduces demand substantially, and total revenue falls. Unitary elasticity is the point of demarcation between the two. At this point, the price increase and the quantity decrease are of identical proportional magnitudes, so that overall expenditure does not alter at all.

9.1.1.1 Monopoly and Elasticity

Observe, we can also express the marginal revenue in terms of the price elasticity of demand

$$MR(q) = p(q)\Big(1 + \frac{1}{e}\Big) \tag{9.1.1}$$

where e is the price elasticity of demand. When demand is inelastic (that is, $e > -1$), MR is negative, when demand is unit elastic ($e = -1$), $MR = 0$, and when demand is elastic ($e < -1$), MR is positive.

Given that the demand curve is downward sloping, $e < 0$. From the expression derived for MR in equation (9.1.1) it then follows that at any positive output level, MR is *smaller than price*, that is, the MR curve lies below the demand curve.

Note that in the competitive case the firm faces an infinitely elastic demand curve, that is, $e \to -\infty$ in this case. Therefore, from equation (9.1.1) it follows that for a competitive firm marginal revenue coincides with price.

9.1.2 The Monopolist's Output Decision

The monopolist chooses quantity to

$$\max_{q \geq 0} \pi(q) \equiv \max_{q \geq 0} [p(q)q - c(q)].$$

The profit maximizing condition is

$$MR(q) = MC(q) \quad \Rightarrow \ p(q)[1 + \frac{1}{e}] = MC(q).$$

If output is such that $MR > MC$, then profits can be increased by increasing output, if output is such that $MR < MC$, then profits can be increased by decreasing output. It is only when $MR = MC$ that profits are maximized.

We have noted that if the demand for the monopolist's output is inelastic ($e > -1$), then marginal revenue is negative. In such a case it is not possible to equate marginal revenue with marginal cost, which is positive. That is, the profit maximizing condition derived above is not applicable. Thus, a profit maximizing monopoly firm will never operate at a price and level of output such that the demand curve is inelastic at that output. If demand is inelastic, then a reduction of output increases the revenue and decreases the cost, hence profits increase. Consequently, a monopoly firm cannot maximize profits at an output where demand is inelastic.

9.1.3 Market Power

Lerner (1934) defined the market power or monopoly power of a firm as the proportional excess of price over marginal cost. Rearranging the monopolist's profit maximization condition we can express the Lerner market power index L for the monopolist as

$$L = \frac{p(q) - MC(q)}{p(q)} = -\frac{1}{e}.$$

The measure L is an indicator of the ability of the firm to set price above the marginal cost. Since the monopolist operates in the region where the demand curve is elastic, $O < L < 1$. Thus, market price is a mark-up over marginal cost for the monopolist.

A zero value of the Lerner market power index for a firm expresses the inability of the firm to influence the price and hence set it above the marginal cost. Clearly, for a competitive firm, market power index will take on the value of zero. A positive value of the Lerner index means that the firm has some market power – it has some ability to influence the price level.

Observe, L depends on the monopolist's optimal output level. If the monopolist faces a constant elasticity demand curve, then L, the proportionate excess of price over marginal cost, becomes a constant.

9.1.4 Linear Demand Curve

Recall the example we considered in Chapter 2: A monopolist faces a linear demand curve $p(q) = a - bq$, where a, $b > 0$. The slope of the demand curve is $-b$. The constant a is the maximum price that the monopolist can charge. The revenue function is

$$R(q) = p(q)q = aq - bq^2.$$

Marginal revenue is $MR = a - 2bq$, it has the same vertical intercept, a, as the demand curve, but it has a horizontal intercept which is half of that of the demand curve. We show this panel (a) of Figure 9.1.

The elasticity of the demand curve is $-\dfrac{a - bq}{bq}$. The point of unitary elasticity occurs at the midpoint of the demand curve, that is, at the output level $\dfrac{a}{2b}$ and marginal revenue is zero at that output. Demand is elastic or inelastic accordingly the output level is less thatn or greater than $\dfrac{a}{2b}$.

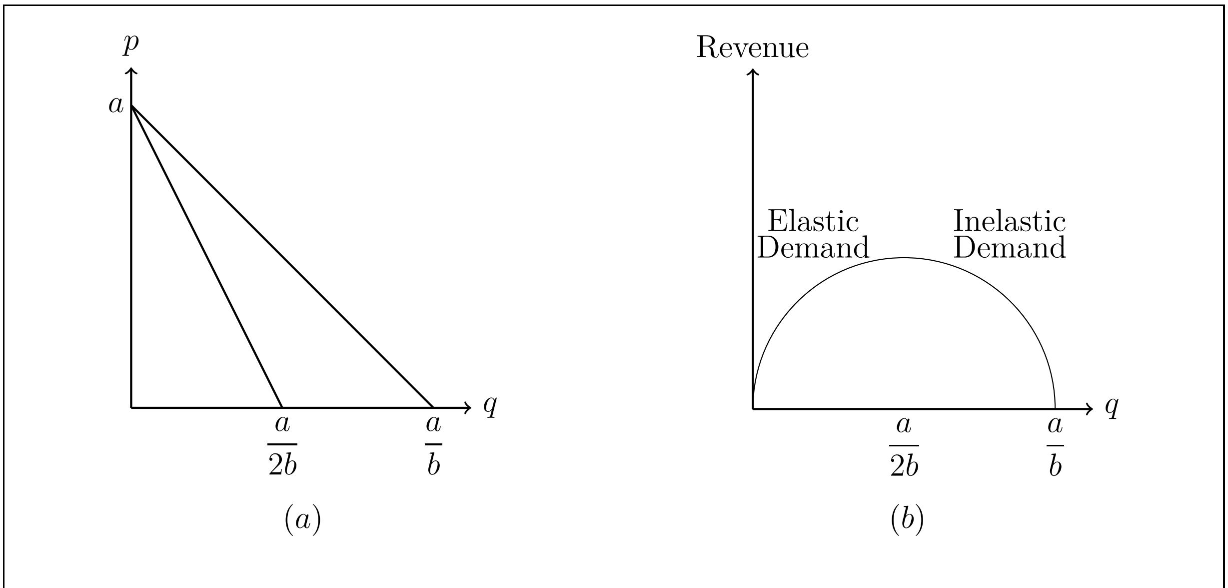


Figure 9.1: Monopoly Revenue, Marginal Revenue: Linear Demand

We know that if the demand for a monopolist's output is elastic (inelastic), then an increase in output leads to an increase (a decrease) in the monopolist's revenue. Therefore as output increases from 0 to $\dfrac{a}{2b}$, where marginal revenue is zero, revenue is at a maximum. Finally, as output increases from $\dfrac{a}{2b}$ to $\dfrac{a}{b}$, revenue starts declining. This is shown in panel (b) of Figure 9.1.

In terms of the revenue graph at any output level, average revenue equals the slope of the line drawn from the origin to the revenue graph (see panel (a) of Figure 9.2). Average revenue at q_1, $AR(q_1) = \dfrac{R(q_1)}{q_1}$= slope of the line drawn from the origin to the $R(q)$ curve at q_1. On the other hand, marginal revenue can be interpreted as the slope of the revenue function. In panel (b) of Figure 9.2, $MR(q_0)$, marginal revenue at q_0, is the slope of the line AB, which is tangent to $R(q)$ at $q = q_0$.

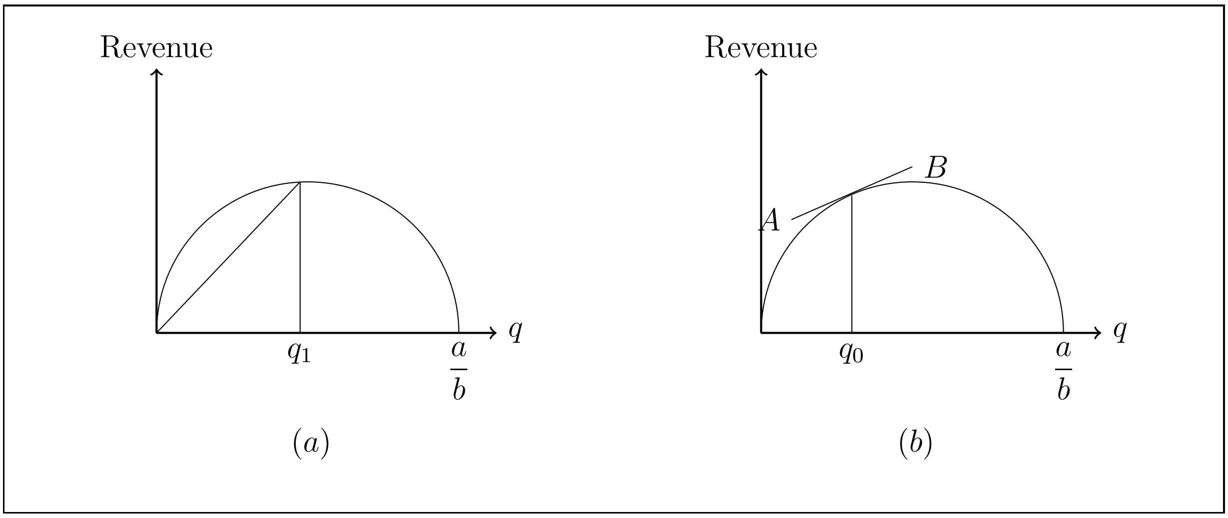


Figure 9.2: Monopoly Average Revenue and Marginal Revenue: Linear Demand

9.1.4.1 Profit Maximization

Let c be the marginal cost. As we have seen in Chapter 2, the problem of the firm is to

$$\max_{q\geq 0} \Pi \equiv p \cdot q - c \cdot q \qquad \Leftrightarrow \max_{q\geq 0} q[a - bq] - cq$$

The first order condition (F.O.C.) is $\dfrac{\partial \Pi}{\partial q} = 0 \quad \Rightarrow \quad a - 2bq - c = 0 \quad \Rightarrow q = \dfrac{a-c}{2b}$.
In order to ensure that this quantity indeed maximizes the firm's profit (and not minimizes it) we need to check the second order condition.
Second order condition (S.O.C.) $\dfrac{\partial^2 \Pi}{\partial q^2} = -2b < 0$ (so S.O.C. is satisfied).
Hence, $q^M = \dfrac{a-c}{2b}$ and $p^M = \dfrac{a+c}{2}$

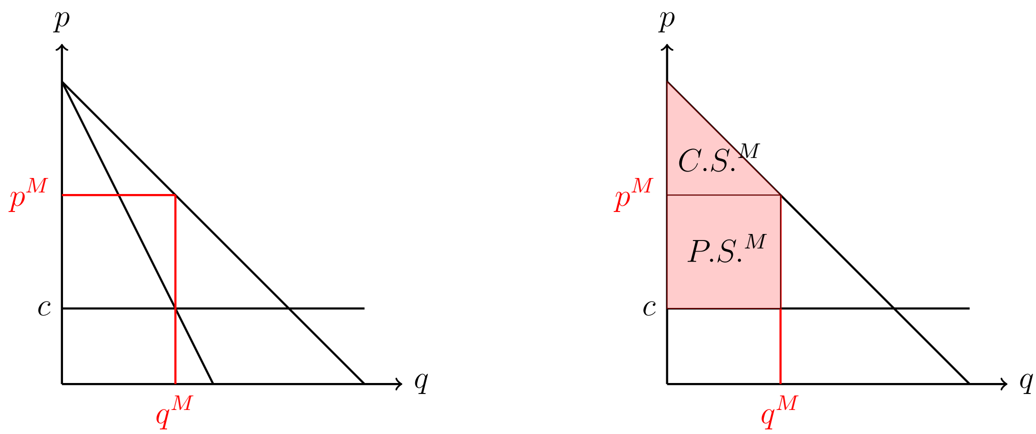
9.1.4.2 Welfare

Consumer surplus is

$$C.S.^M = \frac{1}{2} \cdot (a - p^M) \cdot q^M = \frac{(a-c)^2}{8b}.$$

Producer surplus is

$$P.S.^M = (p^M - c) \cdot q^M = \frac{(a-c)^2}{4b}.$$



$$q^M = \frac{a-c}{2b}, \text{ and } p^M = \frac{a+c}{2}.$$

$$C.S.^M = \frac{(a-c)^2}{8b}, \; P.S.^M = \frac{(a-c)^2}{4b}$$
$$S.W.^M = C.S.^M + P.S.^M = \frac{3(a-b)^2}{8b}.$$

Figure 9.3: Monopoly: Equilibrium Quantity and Price

Social Welfare is
$$S.W.^M = C.S.^M + P.S.^M = \frac{3(a-b)^2}{8b}.$$

9.1.5 Price Discrimination

In some cases a monopolist finds it possible and profitable to sell different units of output at different prices. Selling different units of output at different prices is referred to as price discrimination. Everyday experience confirms that pricing is more complicated than simple uniform price per unit. It should be mentioned that *three conditions are essential for successful price discrimination.* Their presence does not always ensure price discrimination, but it can never happen without them. First, a firm must have some market power, otherwise it can never succeed in charging non-uniform prices. Second, the firm must know the consumer's willingness to pay. Third, a firm must be able to prevent or limit resale from consumers who pay the lower price to those who pay the higher price. In such a case consumers who buy directly from the firm at a higher price will never do that: They will buy at a lower resale price from the group which is charged a lower price by the firm. Economists generally consider these three types of price discrimination: First-degree, Second-degree and Third-degree. We will discuss each of these three kinds of price discrimination separately.

9.1.5.1 First-Degree Price Discrimination

First-degree price discrimination, also called perfect price discrimination occurs when the monopolist is able to sell different units of output at different prices, extracting the highest possible price for each unit sold. Hence under first-degree price discrimination each unit is sold to the individual who values it most highly, that is, who is willing to pay the maximum price for it. Thus, in this case prices may differ from person to person. It is clear that in such a market consumers have no consumer's surplus remaining. (Recall that consumer's surplus is the maximum amount that could be extracted from a consumer above and beyond what he is paying.)

Suppose that the monopolist can identify each consumer and knows the maximum amount that each one is willing to pay. Then the firm will charge each consumer a different price and extract the full consumer's surplus from each one. As long as the price to the last buyer exceeds the marginal cost, the monopolist will sell more of his product to additional consumers, because he makes an extra profit from additional sales. Therefore under first-degree price discrimination the monopolist charges the last buyer the marginal cost. Perfect price discrimination entirely avoids the effect of decreased revenues that result from passing along a lower price to all who were already buying the good, because a low price to one consumer can no longer be passed on to another. Elimination of this effect ensures that the demand curve becomes the marginal revenue curve.

An interesting feature of perfect price discrimination, is that it generates the same output as would be produced under perfect competition. The reason is that under both cases consumers purchase the good as long as they value it above marginal cost. Therefore, under perfect price discrimination the sum of the producer's and consumer's surplus is maximized. However, the producer captures all the surplus generated in the market. Consequently, perfect price discrimination entails no efficiency loss, but alters the distribution of income (more of the consumer's money transferred to the monopolist).

Real world examples of first-degree price discrimination are hard to find. But it generates a resource allocation mechanism other than a perfectly competitive market that achieves Pareto efficiency. Since perfect price discrimination requires detailed knowledge about buyers, it is more likely to occur when one-by-one bargaining occurs. A doctor may be able to successfully price discriminate if he can identify the wealthy people in the area.

9.1.5.2 Second-Degree Price Discrimination

According to second-degree price discrimination, the monopolist sells different units of output at different prices, but every consumer who purchases the same quantity of the good pays the same price. *Thus price differs across units of the good, but not across buyers.* In second-degree price discrimination, total expenditure on the good is not equal to a constant price multiplied by the quantity of the good bought. That is, a consumer's total expenditure on the good does not increase linearly or proportionately with the quantity bought. Therefore second-degree price discrimination can also be called nonlinear pricing.

A simple type of non-linear pricing schedule is a *two-part tariff*. A two-part tariff consists of a lump sum fee plus a charge per unit. For example, in an amusement park, visitors pay an admission fee and then separate fees for each ride. In a club, all members pay a common annual membership fee together with usage fees that vary with the intensity of the use of the club's facilities. Telephone companies charge a subscription fee and then a usage fee that depends on the number of calls made. Many firms that rent out copy machines charge a rental fee plus a fee that depends on the extent of use of the machine. Electricity bills are often computed according to block schedules, in which the first block of units of usage incurs one charge, and subsequent blocks of units incur a different charge.

Here we will analyse a simple case of determination of an optimal two-part tariff. Let us suppose that the consumers are identical and demand more units of the monopolist's good as price falls. That is, each consumer is identical to all other and has the same downward sloping demand curve for the monopolist's good as price falls. Marginal cost is assumed to be constant, say c.

The monopolist could charge a price of c per unit consumed and also charge each (identical) consumer S for the right to buy the product, where S is the common consumer's surplus. A consumer pays the amount S irrespective of the number of units of the product he buys. The monopolist will extract all of the surplus (consumer's surplus plus producer's surplus) generated in this market if he sets a price which is equal to marginal cost. When price equals marginal cost, total surplus is maximized. Therefore, since the monopolist is maximizing the total surplus in this case, the optimal (profit maximizing) two-part tariff is to charge each consumer S for the right to buy and a per unit price c. As stated earlier this pricing method yields competitive output.

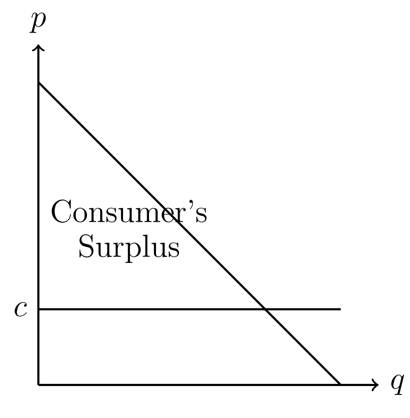


Figure 9.4: Optimal Two-Part Tariff

If the consumers have non-identical downward sloping demand curves, then the monopolist will still charge each consumer c per unit consumed but a different lump sum fee to extract all the consumer's surplus. Therefore, the monopolist must have knowledge about each consumer's demand curve for designing a pricing policy that captures the consumer's surplus of each consumer.

9.1.5.3 Third-Degree Price Discrimination

Third-degree price discrimination is the most common form of price discrimination. It occurs when the monopolist sells output to different persons at different prices, but the price of output sold to a given person is the same for all units. It may arise in the following way. Suppose a firm does not have sufficient information to identify each consumer and is unable to practice first-degree price discrimination. But suppose the firm can determine whether a particular customer belongs to one group or another. If it is possible to prevent resale between the two groups, and if the firm knows the aggregate demand curve of each group, then it becomes profitable to charge different prices to different groups. This is third-degree price discrimination. For instance, magazines charge a higher price to people who buy from newsstands than to those who subscribe. Many cinema halls offer discounts to students. Products are often packed with discount coupons that allow the bearer to purchase the product at a low price the next time.

How does the monopolist determine optimal prices for the two groups of customers? Let q_1 and q_2 be the quantities of the product sold to two groups (markets). Suppose that $p_1(q_1)$ and $p_2(q_2)$ are the prices of the good in the two markets, and that $c(q_1+q_2)$ is the cost of producing the total output (q_1+q_2). Then the monopolist's profits are given by

$$\pi = p_1(q_1)q_1 + p_2(q_2)q_2 - c(q_1+q_2).$$

The first order conditions for profit maximization are

$$MR_1(q_1) = MC_1(q_1+q_2) \text{ and } MR_2(q_2) = MC_2(q_1+q_2)$$

where $MR_i(q_i)$ is the marginal revenue from market i $(i = 1, 2)$ and MC is the marginal cost. Note that since the monopolist produces in a single plant, the marginal cost of production is the same no matter to which market the output is sold. Now,

$$MR_i(q_i) = p_i(q_i)[1 + \frac{1}{e_i}]$$

where e_i is the price elasticity of demand in market$i$. Therefore, we have

$$p_1(q_1)[1+\frac{1}{e_1}] = p_2(q_2)[1+\frac{1}{e_2}] \quad \Rightarrow \quad \frac{p_1(q_1)}{p_2(q_2)} = \frac{1+\frac{1}{e_1}}{1+\frac{1}{e_2}}.$$

So if $e_1 > e_2$ (market 2 has a more elastic demand), p_1 will exceed p_2. That is, the market with lower elasticity of demand will have a higher price. This seems quite reasonable. An elastic demand is a price sensitive demand. A price discriminating firm will set a low price for the price sensitive group and a higher price for the price insensitive group.

Certain conditions are necessary for a monopolist to practise third-degree price discrimination. First, the monopolist must be able to keep the markets apart so that resale is ruled out. Market separation may be assured by high transportation costs. Second, the elasticities of demand at each price level must differ among markets. Third, the sources of price differentiation among the markets must be something other than variation in costs of producing and selling the product in different markets.

Two sources of inefficiency are present in third-degree price discrimination. The first one is the usual one associated with standard monopoly: price exceeds marginal cost. The second is a consumption inefficiency. Since different consumers pay different prices for a product, each consumer's marginal willingness to pay is not the same. *Note that third-degree price discrimination can benefit the society compared to simple monopoly if it produces higher amount of output.*

In Figure 9.5 we present third-degree price discrimination diagrammatically. The monopolist's marginal cost of production is a constant c. D_1 and D_2 are the demand curves in markets 1 and 2. D_2 is more elastic than D_1. Therefore a price discriminator charges a higher price in market 1.

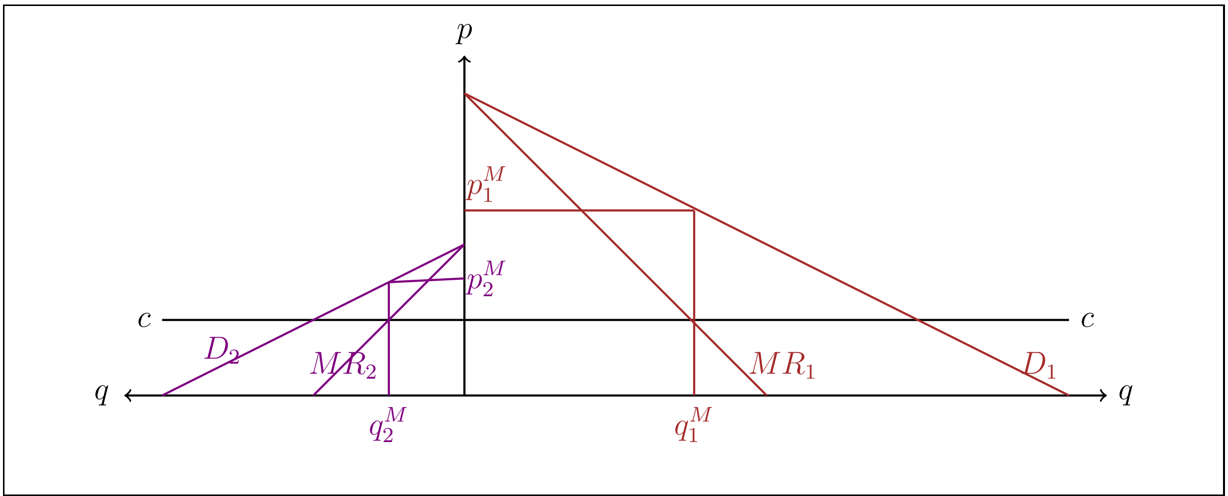


Figure 9.5: Third Degree Price Discrimination

9.2 Oligopoly

9.2.1 Duopoly: Cournot

Recall the example we considered in Chapter 2: There are two firms in the market. Cost functions are the same: marginal cost is c and there is no fixed cost of production. Each firm simultaneously and noncooperatively chooses the "quantity" of production to maximize its profit. Inverse demand function is $p = a - bq = a - b(q_1 + q_2)$, where q_1 and q_2 are quantities chosen by firm 1 and 2.

We know that firm i, where $i = \{1, 2\}$, optimally chooses $q_i^O = \dfrac{1-c}{3b}$. Hence, total output produced is $q^O = \dfrac{2(a-c)}{3b}$ and price is $p^O = \dfrac{a+2c}{3}$.

9.2.2 Welfare:Comparison

We are now in a position to compare optimal quantities and prices under competitive, monopoly and oligopoly market.

$$q^C > q^O > q^M \quad \text{and} \quad p^C < p^O < p^M.$$

We now compare producer's surplus, consumer's surplus and social welfare under competitive, monopoly and oligopoly market.

$$\begin{aligned}
C.S.^M &= \frac{(a-c)^2}{8b} < C.S.^O = \frac{2(a-c)^2}{9b} < C.S.^C = \frac{(a-c)^2}{2b} \\
P.S.^M &= \frac{(a-c)^2}{4b} > P.S.^O = \frac{2(a-c)^2}{9b} > P.S.^C = 0 \\
S.W.^M &= \frac{3(a-c)^2}{8b} < S.W.^O = \frac{4(a-c)^2}{9b} < S.W.^C = \frac{(a-c)^2}{2b}.
\end{aligned}$$

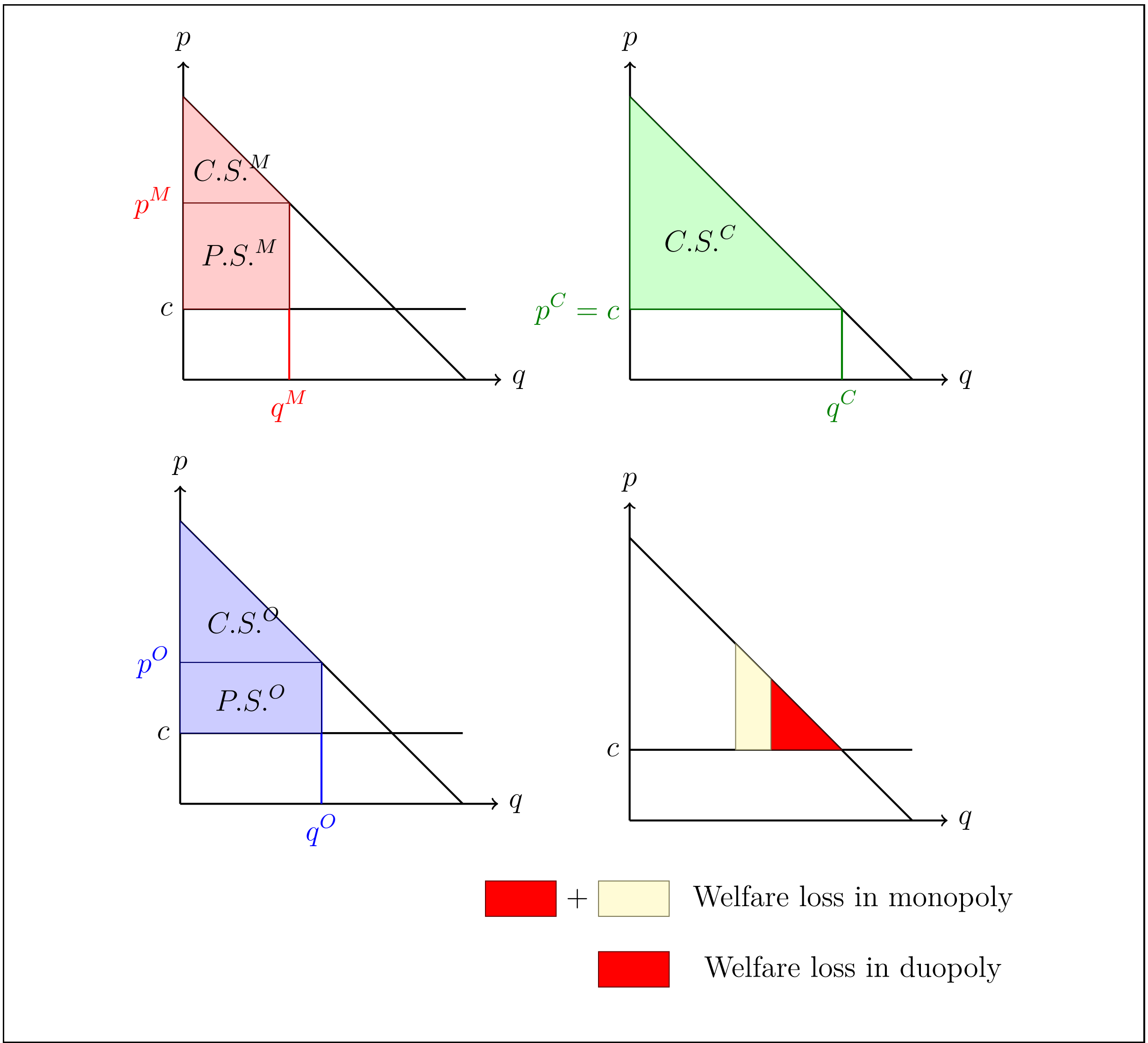


Figure 9.6: Comparison

9.2.3 Cournot and Externality

In any of the oligopoly structures we have considered so far, total profits, in equilibrium, are lower than monopoly profits. This decrease in total profits results from the externality inherent to the process of imperfect competition: When, for example, a firm chooses quantity under Cournot competition, it maximizes its own profit, not taking into account the fact that part of the increase in profits is obtained at the expense of the rival firm's profits.

Let us understand with an example: Suppose there are two firms. The inverse demand function be $p(q)$, where $q(= q_1 + q_2)$ is the total output. And let $c_i(q_i)$ be the total cost of Firm i to produce q_i. Then Firm i's problem is to

$$\max_{q_i \leq 0} \pi_i(q_i, q_j) = q_i p(q) - c_i(q_i).$$

Then the F.O.C. is $q_i p^{'}(q) + p(q) - c_i^{'}(q_i) = 0$.
Now $q_i p^{'}(q)$ captures externality as the firm only take into account the effect of the price change on its own profit. Hence, its output is higher than what would be optimal from the industry's point of view.

Hence, there is a negative externality between Cournot firms. Firms do not internalize the effect that an increase in the quantity they produce has on the other firms. That is, in other words, when Firm i increases q_i it lowers the price to every firm in the market. From the point of view of the industry (i.e. of maximum the total profit) there will be excessive production. To see this, let us consider a general

Exercise 9.1. *Consider the linear demand function $p = a - bq$. Suppose there are two firms, marginal cost of each firm be c. Find out the negative externality imposed by i^{th} firm on the other firm.*

9.2.4 Collusion

It is therefore natural that firms attempt to establish agreements between themselves with a view toward increasing their market power. In fact, it is, in general, possible to find alternative solutions such that all firms are better off (normally at the expense of consumers). This type of behavior is generically designated by collusion.

9.2.4.1 One-Time Interaction and The Stability of Collusive Agreements

Again, consider the linear demand function we considered in section 9.2.1 ($p = a - bq$). Since profit under duopoly is lower than the monopoly, it may seem that duopoly firms would collude. Suppose it is not possible to write legal contracts on quantity to be produced by each firm[1] then *is monopoly quantity sustainable?* Observe the blue line in

[1]In many countries this is illegal.

figure 9.7 plots all the combinations of quantities such that total output is the monopoly amount. Is any point on this line stable? In other words, if both of them agree to produce, say, $(\hat{\alpha}_1, \hat{\alpha}_2)$. Does any firm has an incentive to deviate (break the contract)?

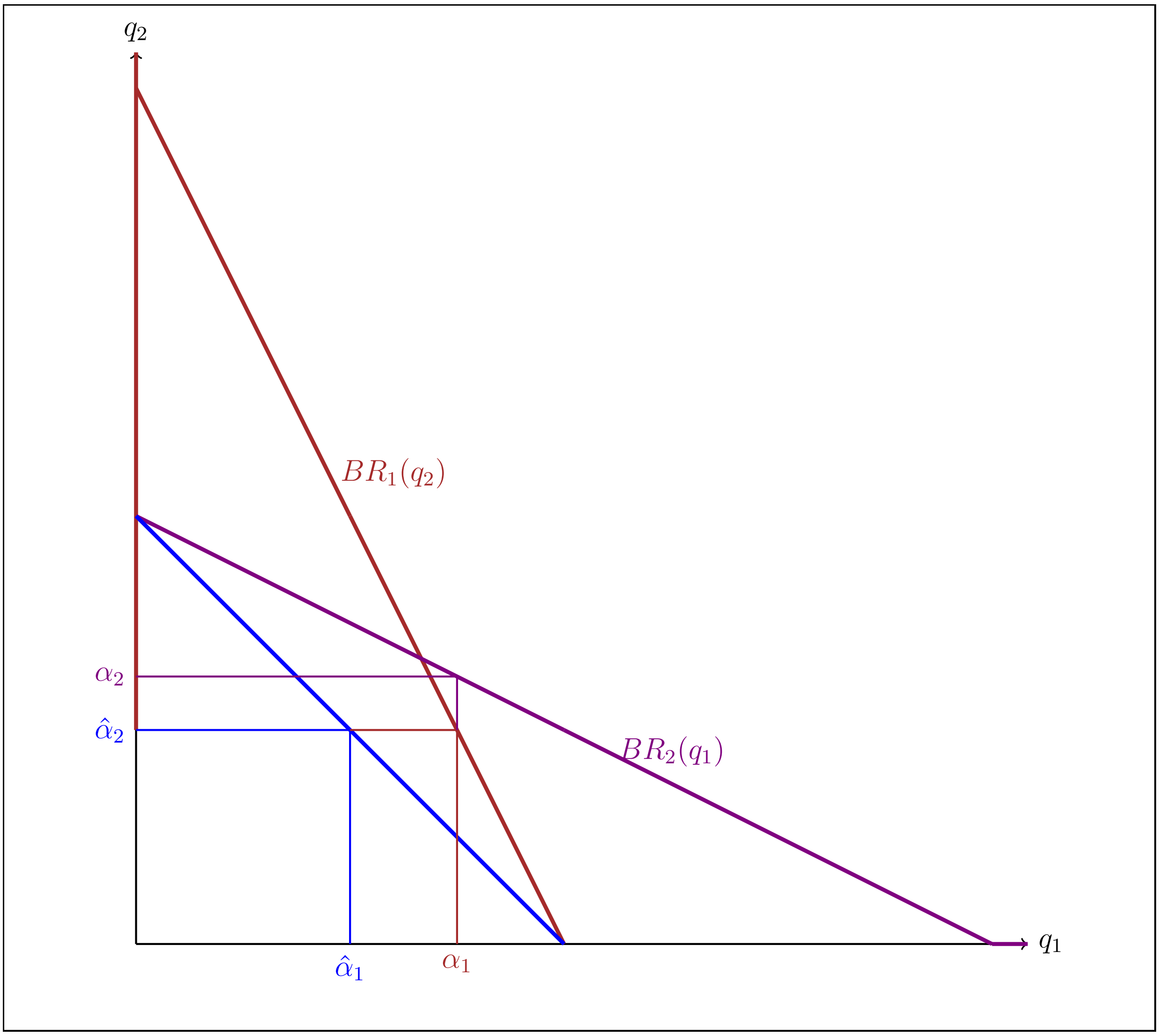


Figure 9.7: Collusion

The answer turns out to be *yes*. To see that consider Firm 1's response, if it believes that Firm 2 would produce $\hat{\alpha}_2$ – it would produce α_1 instead of $\hat{\alpha}_1$. Firm 2 would anticipate this and would produce α_2 instead of $\hat{\alpha}_2$ and so on. Ultimately, they will converge to the Cournot outcome and will produce $\frac{q^O}{2}$.

9.2.4.2 Repeated Interaction and The Stability of Collusive Agreements

For this refer to Chapter 3 section 3.4.

Chapter 10

Information Economics

One of the implicit assumptions of the fundamental welfare theorems is that the characteristics of all commodities are observable to all market participants. Without this condition, distinct markets cannot exist for goods having differing characteristics, and so the complete markets assumption cannot hold. In reality, however, this kind of information is often asymmetrically held by market participants. Consider the following three examples:

1. Recall the market for lemons: The seller of a car has much better information about his car's quality than a prospective buyer does.

2. When a firm hires a worker, the farm may know less than the worker does about the worker's innate ability.

3. When an automobile insurance company insures an individual, the individual may know more than the company about her inherent driving skill and hence about her probability of having an accident.

A number of questions immediately arise about these settings of asymmetric information: How do we characterize market equilibria in the presence of asymmetric information? What are the properties of these equilibria? Are there possibilities for welfare-improving market intervention? In this chapter, we study these questions. For the expositional purposes, we present all the analysis that follows in terms of the labour market example.

10.1 Informational Asymmetries and Adverse Selection

Consider the following simple labor market model adapted from Akerlof's (1970) pioneering work[1]: There are many identical potential firms that can hire workers. Each produces the same output using an identical constant returns to scale technology in which labour

[1] The market for lemons which we have studied in Chapter 4.

is the only input. The firms are risk neutral, seek to maximize their expected profits, and act as price takers. For simplicity, we take the price of the firms' output to equal 1 (in units of a numeraire good).

Workers differ in the number of units of output they produce if hired by a firm, which we denote by θ. Let there be two types of workers – low productive worker θ_L and high productive worker θ_H, where $\theta_H > \theta_L$. Let λ be the proportion of high type workers. Let the total number of workers be N.

Workers seek to maximize the amount that they earn from their labour. A worker can be self-employed or work at a firm. Let worker of type i's productivity at home be $r(\theta_i)$. Thus, $r(\theta_i)$ is the opportunity cost to worker of type θ_i of accepting employment: He will accept employment at a firm if and only if he receives a wage of at least $r(\theta_i)$ (for convenience, we assume that he accepts the offer if he is indifferent). Let us assume that workers are more productive while they work for a firm, that is $\theta_H > r(\theta_H)$ and $\theta_L > r(\theta_L)$.

10.1.1 Benchmark Case: Productivity is Observable

Since the labour of each different type of worker is a distinct good, there is a distinct equilibrium wage for each type. Given the competitive, constant returns nature of the firms, in a competitive equilibrium we have $w^*(\theta_H) = \theta_H$ and $w^*(\theta_L) = \theta_L$ (recall that the price of their output is l), and both types of workers work for the firm (as $w^*(\theta_H) = \theta_H > r(\theta_H)$ and $w^*(\theta_L) = \theta_L > r(\theta_L)$.

As would be expected from the first fundamental welfare theorem, this competitive outcome is Pareto optimal. To verify this, recall that any Pareto optimal allocation of labour must maximize aggregate surplus and that happens in our model when both types of workers work for the firm. Now let us consider the case of asymmetric information: A worker knows his productivity, and the firms only know the proportion of high productive to low productive workers.

10.1.2 Asymmetric Information: Productivity is Not Observable

Since workers' types are not observable, the wage rate must be independent of workers' types, and so we will have a single wage rate w for all the workers.

Supply of labour as a function of wage rate w. A high type worker will be willing to work for a firm if and only if $w \geq r(\theta_H)$. Similarly, a low type worker will be willing to work for a firm if and only if $w \geq r(\theta_L)$. Observe, if a high type works for a firm at an ongoing wage, then low type will also work.

Consider, next, the demand for labour as a function of w. If a firm believes that both types of workers work at w, then the average productivity of worker is $\lambda\theta_H + (1 - \lambda)\theta_L$. Given competition among firms w, in this case, will be $\lambda\theta_H+(1-\lambda)\theta_L$. On the other hand, if w is lower than $r(\theta_H)$, then only low type workers will work and given competition w will be equal to θ_L.

So, at equilibrium when $w \geq r(\theta_H)$ $w = E(\theta) = \lambda\theta_H + (1-\lambda)\theta_L$, otherwise $w = \theta_L$. Observe, this involves *rational expectation* on the part of the firms. That is, firms correctly anticipate the average productivity of these workers who accept employment in the equilibrium.

Hence, as we have seen earlier (in Lemons market), the competitive equilibrium will fail to be Pareto optimal, if $\lambda < \dfrac{r(\theta_H) - \theta_L}{\theta_H - \theta_L}$. (Why?)

Now let us assume that $r(\theta_H) = r(\theta_L) = 0$, then the competitive equilibrium will be Pareto efficient (why?). But the high productive workers get $E(\theta)$ which is less than their productivity. This type of workers, thus, have an interest to convince the firms that they deserve more. However, since the low type workers could also pretend to be high type workers and the firms have no way of checking this, more information is necessary to distinguish between them. Since, both the firms and the high-ability workers have incentives to try to distinguish among workers, one might expect such mechanisms to develop in the marketplace. Next, we study two such mechanisms.

10.1.3 Signalling

The mechanism that we examine in this section is that of signaling which was first investigated by Spence (1973, 1974). The basic idea is that high ability workers may have actions they can take to distinguish themselves from their low-ability counterparts.

The simplest example of such a signal occurs when workers can submit to some costless test that reliably reveals their type. It is relatively straightforward to show that in any subgame perfect Nash equilibrium all high ability workers will submit to the test and the market will achieve the full information outcome. Any worker who chooses not to take the test will be correctly treated as low type worker.

However, in many instances, no procedure exists that directly reveals a worker's type. Nevertheless, as the analysis in this section reveals, the potential for signaling may still exist.

Consider the model we considered above, where there two types of workers (high ability θ_H and low ability θ_L) and their reservation wages are zero. The important extension, we introduce here, is that

- Before entering the job market a worker can get some education.
- The amount of education that a worker receives is observable.
- To make matters particularly stark, we assume that education does nothing for a worker's productivity.

- The cost (the cost may be of either monetary or psychic origin) of obtaining education level e for high-ability worker (θ_H) is given by the twice continuously differentiable function $c(e, \theta_H)$, with
  - $*$ $c(0, \theta_H) = 0$: that is if a worker is not getting any education, he does not incur any cost,
  - $*$ $c(e, \theta_H) > 0$: education is costly,
  - $*$ $c_e(e, \theta_H) = \dfrac{\partial c(e, \theta_H)}{\partial e} > 0$: marginal cost of education is positive, that is cost increases with education and
  - $*$ $c_{ee}(e, \theta_H) = \dfrac{\partial^2 c(e, \theta_H)}{\partial e^2} > 0$: cost increases with education at an increasing rate.
- We make similar assumption for low ability worker (θ_L):
  - $*$ $c(0, \theta_L) = 0$,
  - $*$ $c(e, \theta_L) > 0$,
  - $*$ $c_e(e, \theta_L) = \dfrac{\partial c(e, \theta_L)}{\partial e} > 0$ and
  - $*$ $c_{ee}(e, \theta_L) = \dfrac{\partial^2 c(e, \theta_L)}{\partial e^2} > 0$.
- We also assume that both cost and the marginal cost of education are lower for high-ability workers; for example, the work required to obtain a degree might be easier for a high-ability individual:
  - $*$ $c(e, \theta_H) < c(e, \theta_L)$ and
  - $*$ $c_e(e, \theta_H) < c_e(e, \theta_L)$.
- Let $u(w, e|\theta_H)$ denote the utility of a type θ_H worker who chooses education level e and receives wage w, we take $u(w, e|\theta_H)$ to equal her wage less any educational costs incurred: $u(w, e|\theta_H) = w - c(e, \theta_H)$. Recall, we have assumed that his reservation wage is 0.

  Similarly, low-ability worker's utility when he earns wage w chooses education level e is $u(w, e|\theta_L) = w - c(e, \theta_L)$.

  In Figure 10.1 we draw indifference curves (ICs) of two different types of workers (with wages measured on the vertical axis and education levels measured on the horizontal axis). Let us recall how to find the slope of an IC for a high type worker

$$u(w, e|\theta_H) = w - c(e, \theta_H)$$

  Totally differentiating we get

$$0 = dw - \frac{\partial c(e, \theta_H)}{\partial e} de \quad \Rightarrow \quad \frac{dw}{de}\Big|_{\bar{u}} = \frac{\partial c(e, \theta_H)}{\partial e} = c_e(e, \theta_H).$$

Similarly, slope of IC of low type worker is $\frac{dw}{de}\Big|_{\bar{u}} = \frac{\partial c(e,\theta_L)}{\partial e} = c_e(e,\theta_L)$.
Note that, ICs are upward sloping and the slope of an IC of high type worker is smaller than that of low type worker. Hence, they can cross only once. This property of preference is known as the *single-crossing property.* It arises because the high type worker's marginal rate of substitution between wages and education at any given (w, e) pair is lower than that of low type worker:

$$
\begin{aligned}
\text{Slope of IC of high type} &= \frac{dw}{de}\Big|_{\bar{u}} \\
&= c_e(e,\theta_H) \\
&< c_e(e,\theta_L) \\
&= \frac{dw}{de}\Big|_{\bar{u}} \\
&= \text{Slope of IC of low type}
\end{aligned}
$$

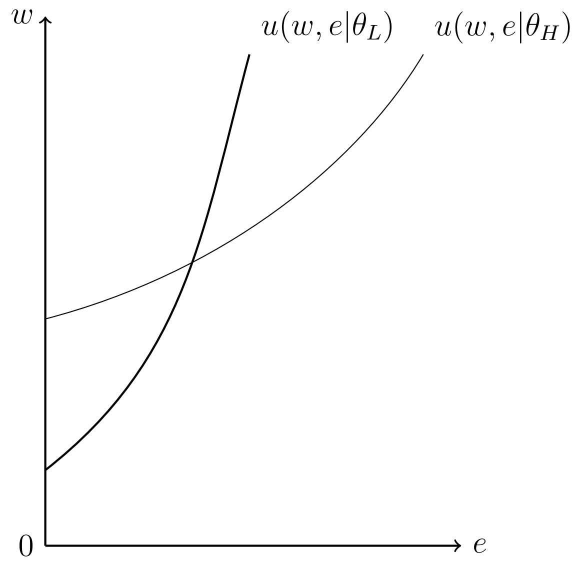


Figure 10.1: Indifference Curves

In the analysis that follows, we shall see that this otherwise useless education may serve as a signal of unobservable worker productivity. In particular, equilibria emerge in which high-productivity workers choose to get more education than low-productivity workers and firms correctly take differences in education levels as a signal of ability. The welfare effects of signaling activities are generally ambiguous. By revealing information about worker types, signaling can lead to a more efficient allocation of workers' labour and in some instances to a Pareto improvement. At the same time *because signaling activity is costly*, workers' welfare may be reduced if they are compelled to engage in a high level of signaling activity to distinguish themselves.

We are now ready to determine the equilibrium education choices for the two types of workers. It is useful to consider separately two different types of equilibria that might arise: *Separating equilibria*, in which the two types of workers choose different education levels, and *pooling equilibria*, in which the two types choose the same education level.

10.1.3.1 Separating Equilibria

Let, $e^*(\theta_H)$ and $e^*(\theta_L)$ be the education chosen by high and low type workers respectively. After observing these education level, firms offer wages. Let $w^*(e)$ be the firm's equilibrium wage offer as a function of the worker's education level. We want to find out the equilibria, that is equilibrium value of $e^*(\theta_H)$, $e^*(\theta_L)$, $w^*(e^*(\theta_H))$, and $w^*(e^*(\theta_L))$. We first establish two useful lemmas.

Lemma 10.1. *In any separating equilibrium,* $w^*(e^*(\theta_H)) = \theta_H$ *and* $w^*(e^*(\theta_L)) = \theta_L$*; that is, each worker type receives a wage equal to her productivity level.*

Proof. Beliefs on the equilibrium path must be correct. Hence, upon seeing education level $e^*(\theta_H)$ firms must assign probability one to the worker being type θ_H. Likewise. upon seeing education level $e^*(\theta_L)$, firms must assign probability one to the worker being type θ_L. Now, the firms are competitive, hence, the resulting wages are θ_H and θ_L. ■

Lemma 10.2. *In any separating equilibrium,* $e^*(\theta_L) = 0$*; that is, a low-ability worker chooses to get no education.*

Proof. Suppose not, that is, that when the worker is type θ_L, she chooses some strictly positive education level $\hat{e} > 0$. According to the lemma above, by doing so, the worker receives a wage equal to θ_L. However, she would receive a wage of at least θ_L, if she instead chose $e = 0$. Since choosing $e = 0$ would have save her the cost of education, she would be strictly better off by doing so, which is a contradiction to the assumption that $\hat{e} > 0$ is her equilibrium education level. ■

This lemma implies that, in any separating equilibrium, type θ_L's indifference curve through his equilibrium level of education and wage must look as depicted in Figure 10.2.

Now we want to find out $e^*(\theta_H)$. But before that, what is the belief of a firm about a worker's type upon observing education level e? Since, at equilibrium low type chooses $e^*(\theta_L)$ and high type chooses $e^*(\theta_H)$, and at equilibrium beliefs must be correct, whenever a firm observes $e^*(\theta_L)$ it believes that the worker is of low type and whenever it observes $e^*(\theta_H)$ it believes that the worker's type is high. But what about when it observes any e which is neither $e^*(\theta_L)$ not $e^*(\theta_H)$? Since, at equilibrium all low type workers would choose $e^*(\theta_L)$ and high type workers would choose $e^*(\theta_H)$, such an e will not be observed at the equilibrium. But, we need to specify the beliefs and hence, wage rate for all possible beliefs as that will give us the payoff from deviation (by a worker). These beliefs are known as *off the equilibrium path.* There could be many such *off the equilibrium path* belief, one of them is whenever a firm observes any education level below $e^*(\theta_H)$, it

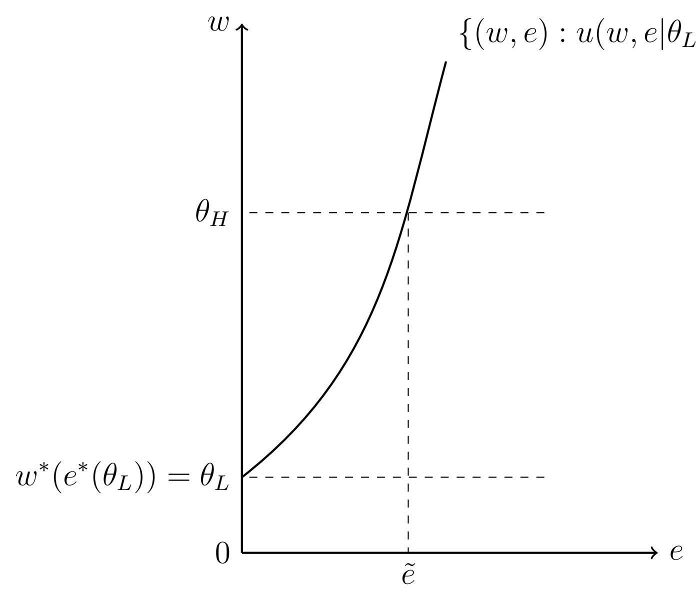


Figure 10.2: Low-ability Worker's Outcome in A Separating Equilibrium

believes that the worker is of low type and whenever it observes education level above $e^*(\theta_H)$, it believes that the worker is of high type.

Now we want to find out $e^*(\theta_H)$. Many education levels for the high-ability type are possible. Next, we will find the range of that: We will see that any education level between $\tilde{e}$ and $\bar{e}$ in Figure 10.3 can be the equilibrium education level of the high-ability workers. How are we getting $\tilde{e}$ and $\bar{e}$?

The education level of the high-ability worker cannot be below $\tilde{e}$ in a separating equilibrium because, if it were, the low-ability worker would deviate and pretend to be of high ability by choosing the high-ability education level. (In Figure 10.3, see low type worker prefers (θ_H, e_1) over $(\theta_L, 0)$. So, if $e^*(\theta_H) = e_1$, then a low-ability worker would get that much education in order to pretend to be a high type. So, it will not be a separating equilibrium).

On the other hand, the education level of the high-ability worker cannot be above $\bar{e}$ because, if it were the high-ability worker would prefer to get no education, even if this resulted in him being thought to be of low ability. (In Figure 10.3, see high type worker prefers $(\theta_L, 0)$ over (θ_H, e_2). So, if $e^*(\theta_H) = e_2$, then a high-ability worker would prefer to net get any education even though that means he will earn only θ_L).

Note that these various separating equilibria can be Pareto ranked. In all of them firms earn zero profits, and a low-ability worker's utility is θ_L. However, a high-ability worker does strictly better in equilibria in which she gets a lower level of education.

Thus separating equilibria in which the high-ability worker gets education level $\tilde{e}$ Pareto dominate all the others. The Pareto dominated equilibria are sustained because of the high-ability worker's fear that if he chooses a lower level of education than that prescribed in the equilibrium firms will believe that he is not a high-ability worker. These

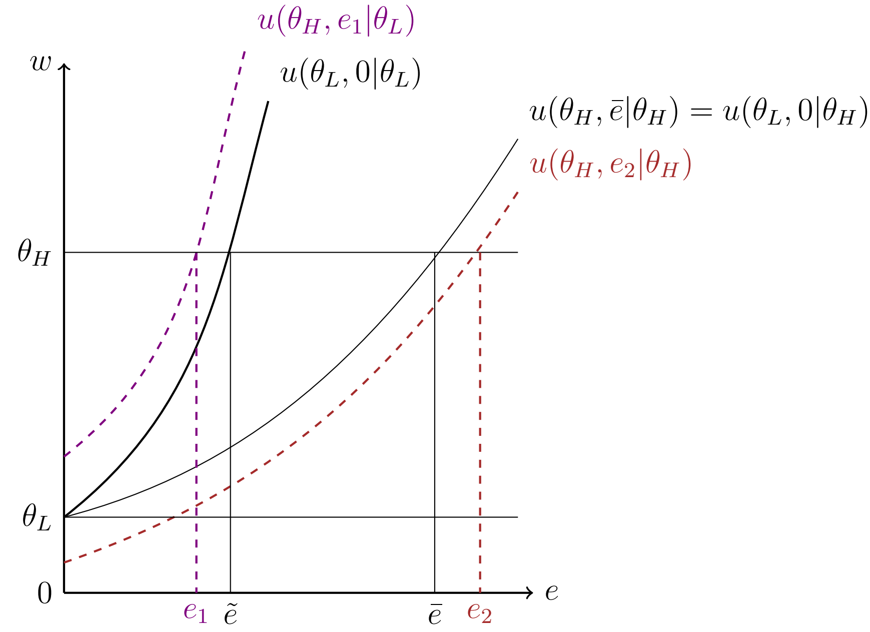


Figure 10.3: Separating Equilibria: Choice of Education of High-ability Type Worker

beliefs can be maintained because in equilibrium they are never disconfirmed.

It is of interest to compare welfare in these equilibria with that arising when worker types are unobservable but no opportunity for signaling is available. When education is not available as a signal (so workers also incur no education costs). In both cases, firms earn expected profits of zero However low-ability workers are strictly worse off when signaling is possible: In both cases they incur no education costs, but when signaling is possible they receive a wage of θ_L rather than $E(\theta)$ (recall $E(\theta) = \lambda\theta_H + (1-\lambda)\theta_L > \theta_L$ whenever $\lambda > 0$ and $\theta_H > \theta_L$).

What about high-ability workers? The somewhat surprising answer is that high-ability workers may be either better or worse off when signaling is possible. In Figure 10.4 (a) the high-ability workers are better off because of the increase in their wages arising through signaling. However, in Figure Figure 10.4 (b), even though high-ability workers seek to take advantage of the signaling mechanism to distinguish themselves, they are *worse off* than when signaling is impossible! Although this may seem paradoxical (if high-ability workers choose to signal, how can they be worse off?), its cause lies in the fact that in a separating signaling equilibrium firms' expectations are such that the wage education outcome from the no-signaling situation $(w, e = (E(\theta), 0)$, is no longer available to the high-ability workers; if they get no education in the separating signaling equilibrium, they are thought to be of low ability and offered a wage of θ_L. Thus, they can be worse off when signaling is possible, even though they are choosing to signal.

Note that because the set of separating equilibria is completely unaffected by the fraction λ of high-ability workers, as this fraction grows it becomes more likely that the high-ability workers are made worse off by the possibility of signaling (Figure 10.4 (a)

and (b)). In fact, as this fraction gets close to 1, nearly every worker is getting costly education just to avoid being thought to be one of the handful of bad workers!

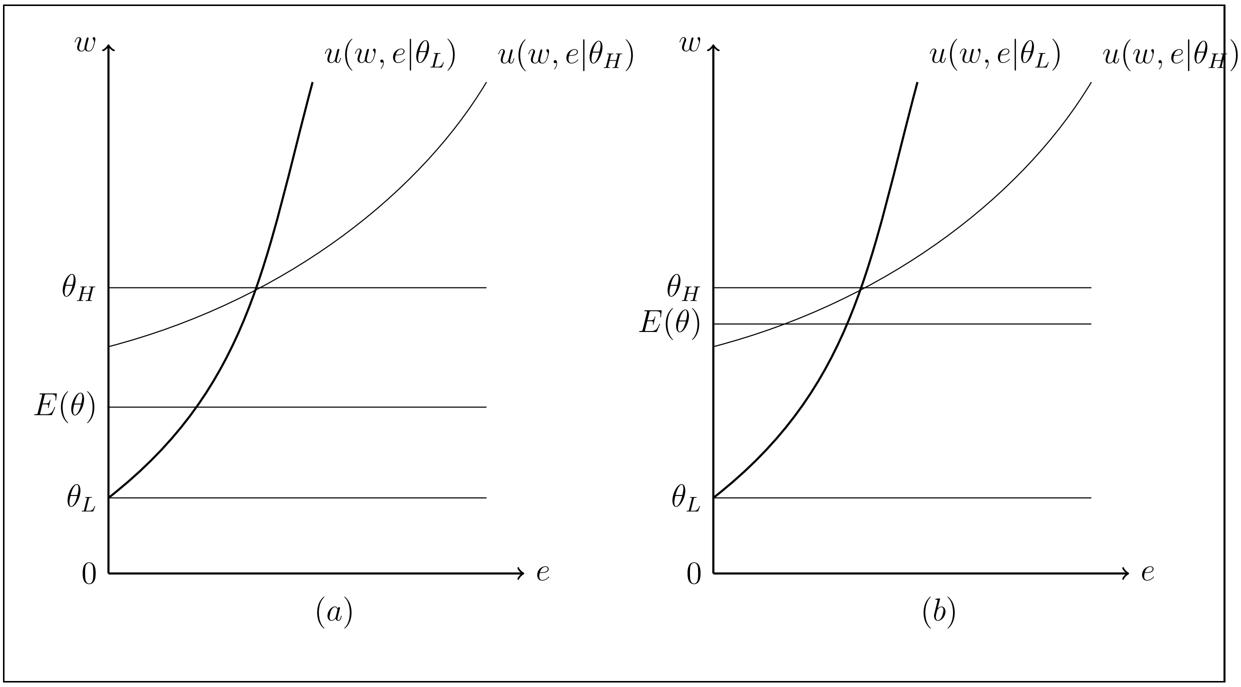


Figure 10.4: Separating Equilibria may be Pareto Dominated by the No-signalling Outcome

10.1.3.2 Pooling Equilibria

Consider now pooling equilibria, in which the two types of workers choose the same level of education, $e^*(\theta_L) = e^*(\theta_H) = e^*$. Since the firms beliefs must be correct: Their beliefs when they see education level e^* must assign probability λ to the worker being type θ_H. Thus, in any pooling equilibrium, we must have $w^*(e^*) = \lambda\theta_H + (1-\lambda)\theta_L = E(\theta)$.

The only remaining issue therefore concerns what levels of education can arise in a pooling equilibrium. It turns out that any education level between 0 and the level $\underline{e}$ depicted in Figure 10.5 can be sustained. Observe, education levels greater than $\underline{e}$ cannot be sustained because a low-ability worker would rather set a $e = 0$ than $e > \underline{e}$ even if this results in a wage payment of θ_L.

Note that a pooling equilibrium in which both types of worker get no education Pareto dominates any pooling equilibrium with a positive education level. Once again, the Pareto-dominated pooling equilibria are sustained by the worker's fear that a deviation will lead firms to have an unfavorable impression of his ability. Note also that a pooling equilibrium in which both types of worker obtain no education results in exactly the same outcome as that which arises in the absence of an ability to signal. Thus. pooling equilibria are (weakly) Pareto dominated by the no-signaling outcome.

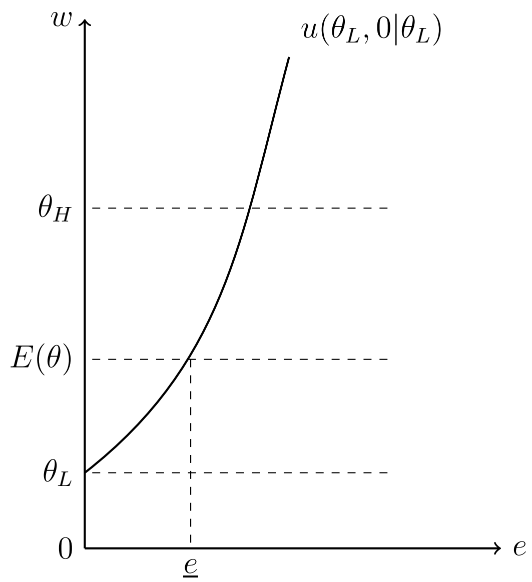


Figure 10.5: The Highest-Possible Education Level in a Pooling Equilibrium

10.2 The Principal-Agent Problem: Moral Hazard

In Section 10.1 we considered situations in which asymmetries of information exist between individuals at the time of contracting. Now we shift our attention to asymmetries of information that develop *subsequent to the signing of a contract.*

Even when informational asymmetries do not exist at the time of contracting, the parties to a contract often anticipate that asymmetries will develop sometime after the contract is signed. For example, after an owner of a firm hires a manager, the owner may be unable to observe how much effort the manager puts into the job. Similarly. the manager will often end up having better information than the owner about the opportunities available to the firm.

Anticipating the development of such informational asymmetries, the contracting parties seek to design a contract that mitigates the difficulties they cause. These problems are endemic to situations in which one individual hires another to take some action for him as his "agent". For this reason, this contract design problem has come to be known as the principal-agent problem.

The literature has traditionally distinguished between two types of informational problems that can arise in these settings: Those resulting from hidden actions and those resulting from hidden information. The hidden action case, also known as *moral hazard*, is illustrated by the owner's inability to observe how hard his manager is working; the manager's coming to possess superior information about the firm's opportunities, on the other hand, is an example of hidden information.[2]

[2]The term originates in the insurance literature, which first focused attention on two types of informational imperfections: the "moral hazard" that arises when an insurance company cannot observe whether the insured exerts effort to prevent a loss and the "adverse selection" that occurs when the insured knows

It is important to emphasize the broad range of economic relationships that fit into the general framework of the principal-agent problem. The owner-manager relationship is only one example; others include insurance companies and insured individuals (the insurance company cannot observe how much care is exercised by the insured), manufacturers and their distributors (the manufacturer may not be able to observe the market conditions faced by the distributor), a firm and its workforce (the firm may have more information than its workers about the true state of demand for its products and therefore about the value of the workers' product), and banks and borrowers (the bank may have difficulty observing whether the borrower uses the loaned funds for the purpose for which the loan was granted). As would be expected given this diversity of examples, the principal-agent framework has found application in a broad range of applied fields in economics.

Our main focus is to study a seminal paper by *Carl Shapiro and Joseph E. Stiglitz*, *Equilibrium Unemployment as a Worker Discipline Device*, published in *The American Economic Review*, (Volume 74, No. 3, pp. 433-444) in June, 1984. But before that, let us consider a simple model.

10.2.1 Incentives in the Principal–Agent Framework

A small manufacturer uses labour and machinery to produce watches. The owners want to maximize profit. They must rely on a machine repair-person whose effort will influence the likelihood that machines break down and thus affect the firm's profit level. Revenue also depends on other random factors, such as the quality of parts and the reliability of other labour. As a result of high monitoring costs, the owners can neither measure the effort of the repair-person directly nor be sure that the same effort will always generate the same profit level. Table 10.6 describes these circumstances.

	Bad Luck	Good Luck
Low Effort ($e = 0$)	$\$10,000$	$\$20,000$
High Effort ($e = 1$)	$\$20,000$	$\$40,000$

Figure 10.6: Revenue From Making Watches

The table shows that the repair-person can work with either a low or high amount of effort. Low effort ($e = 0$) generates either $\$10,000$ or $\$20,000$ in revenue (with equal probability), depending on the random factors that we mentioned. We label the lower of the two revenue levels "bad luck" and the higher level "good luck". When the repair-person makes a high effort ($e = 1$), revenue will be either $\$20,000$ (bad luck) or $\$40,000$ (good luck). These numbers highlight the problem of incomplete information: When the

more than the company at the time he purchases a policy about his likelihood of an accident.

firm's revenue is $\$20,000$, the owners cannot know whether the repair-person has made a low or high effort.

Suppose the repair-person's goal is to maximize his wage payment less the cost (in terms of lost leisure and unpleasant work time) of the effort that he makes. To simplify, we suppose that the cost of effort is 0 for low effort and $\$10,000$ for high effort. (Formally, $c = \$10,000e$.)

Now we can state the principal–agent problem from the owners' perspective. The owners' goal is to maximize expected profit, given the uncertainty of outcomes and given the fact that the repair-person's behaviour cannot be monitored. The owners can contract to pay the repair-person for his work, but the payment scheme must be based entirely on the measurable output of the manufacturing process, not on the repair-person's effort. To signify this link, we describe the payment scheme as $w(R)$, stressing that *payments can depend only on measured revenue.*

What is the best payment scheme? And can that scheme be as effective as one based on effort rather than output?

The best payment scheme depends on the nature of production, the degree of uncertainty, and the objectives of both owners and managers. The arrangement will not always be as effective as an ideal scheme directly tied to effort. A lack of information can lower economic efficiency because both the owners' revenue and the repair-person's payment may fall at the same time.

Let's see how to design a payment scheme when the repair-person wishes to maximize his payment received net of the cost of effort made. Suppose first that the owners offer a fixed wage payment $\bar{w}$. So, no matter what, the repair person will make $\bar{w}$. Now, if he puts effort, his utility is $\bar{w} - 10,000$, on the other hand, if he does not put any effort, then his utility is $\bar{w}$. Hence, he does not put effort and therefore, a fixed payment will lead to an inefficient outcome.

When $e = 0$ and $w = 0$, the owner will earn an expected revenue of $\$15,000$ and the repair-person a net wage of 0. Both the owners and the repair-person will be better off if the repair-person is rewarded for his productive effort. Suppose, for example, that the owners offer the repair-person the following payment scheme:

$$\text{If } R = \$10,000 \text{ or } \$20,000, \ w = 0,$$
$$\text{If } R = \$40,000, \ w = \$24,000.$$

Under this bonus arrangement, a low effort generates no payment. A high effort, however, generates an expected payment of $\$12,000$, and an expected payment less the cost of effort of $\$12,000 - \$10,000 = \$2000$. Under this system, the repair-person will choose to make a high level of effort. This arrangement makes the owners better off than before because they get an expected revenue of $\$30,000$ and an expected profit of $\$18,000$.

This is not the only payment scheme that will work for the owners, however. Suppose they contract to have the worker participate in the following revenue-sharing arrangement. When revenues are greater than $\$18,000$,

$$w = R - \$18,000$$

(Otherwise the wage is zero.)

In this case, if the repair-person makes a low effort, he receives an expected payment of \$1000. But if he makes a high level of effort, his expected payment is $\$12,000$, and his expected payment less the $\$10,000$ cost of effort is \$2000. (The owners' profit is $\$18,000$, as before.)

Thus, in the example above, a revenue-sharing arrangement achieves the same outcome as a bonus-payment system. In more complex situations, the incentive effects of the two types of arrangements will differ. However, the basic idea illustrated here applies to all principal–agent problems: When it is impossible to measure effort directly, an incentive structure that rewards the outcome of high levels of effort can induce agents to aim for the goals that the owners set.

10.2.2 Equilibrium Unemployment as a Worker Discipline Device: Shapiro-Stiglitz

Involuntary unemployment appears to be a persistent feature of many labour markets. The presence of such unemployment raises the question of why wages do not fall to clear labor markets. In this paper it has been shown how the information structure of employer-employee relationships, in particular the inability of employers to costlessly observe workers' on-the-job effort, can explain involuntary unemployment' as an equilibrium phenomenon. In fact, it has been shown that imperfect monitoring necessitates unemployment in equilibrium.

The intuition behind the result is simple: Under the conventional competitive paradigm, in which all workers receive the market wage and there is no unemployment, the worst that can happen to a worker who shirks on the job is that he is fired. Since he can immediately be rehired, however, he pays no penalty for his misdemeanor. With imperfect monitoring and full employment, therefore, workers will choose to shirk.

To induce its workers not to shirk, the firm attempts to pay more than the "going wage"; then, if a worker is caught shirking and is fired, he will pay a penalty. If it pays one firm to raise its wage, however, it will pay all firms to raise their wages. When they all raise their wages, the incentive not to shirk again disappears. But as all firms raise their wages, their demand for labor decreases, and unemployment results. With unemployment, even if all firms pay the same wages, a worker has an incentive not to shirk. For, if he is fired an individual will not immediately obtain another job. The equilibrium unemployment rate must be sufficiently large that it pays workers to work rather than to take the risk of being caught shirking.

The idea that the threat of firing a worker is a method of discipline is not novel

We consider the following simple model.

10.2.2.1 Workers

There are a fixed number, N, of identical, risk neutral workers, all of whom dislike putting forth effort, but enjoy consuming goods. A worker's utility function is given by $u(w, e) = w - e$, where w is the wage received and e is the level of effort on the job. For simplicity, we assume that workers can provide either minimal effort ($e = 0$), or some fixed positive level of $e > 0$, that is $e \in \{0, e\}$. When a worker is unemployed, he receives unemployment benefits of $\bar{w}$ (and $e = 0$).

Each worker is in one of two states at any point in time: employed or unemployed. There is a probability b that at each period a worker will be separated from his job due to relocation, etc., which will be taken as exogenous. Exogenous separations cause a worker to enter the unemployment pool. Workers maximize the expected present discounted value of utility with a discount rate $\delta > 0$. The model is set in continuous time.

10.2.2.2 The Effort Decision of a Worker

The only choice workers make is the selection of an effort level. If a worker performs at the customary level of effort for his job, that is, if he does not shirk, he receives a wage of w and will retain his job until exogenous factors cause a separation to occur. If he shirks, there is some probability q (exogenous), that he will be caught. If he is caught shirking he will be fired, and forced to enter the unemployment pool. So, the probability with which an employed worker becomes unemployed is $b + q$ if he shirks and $b + q$ if he shirks. While unemployed he receives unemployment compensation of $\bar{w}$ (also discussed below).

The worker selects an effort level to maximize his discounted utility stream. This involves comparison of the utility from shirking with the utility from not shirking, to which we now turn. We define V_E^S, as the expected lifetime utility of an employed shirker, V_E^N as the expected lifetime an employed nonshirker, and V_u as the expected lifetime utility of an unemployed individual. Observe, we can write V_E^S as[3]

$$\begin{aligned}
& V_E^S = \delta\Big[w + (b+q)V_u + (1-(b+q))V_E^S\Big] \\
\Rightarrow\;& (1-\delta)V_E^S = \delta\Big[w + (b+q)[V_u - V_E^S]\Big] \\
\Rightarrow\;& (\frac{1}{\delta} - 1)V_E^S = w + (b+q)[V_u - V_E^S] \\
\Rightarrow\;& rV_E^S = w + (b+q)[V_u - V_E^S] && (10.2.1) \\
\Rightarrow\;& V_E^S = \frac{w + (b+q)V_u}{r+b+q}. && (10.2.2)
\end{aligned}$$

where $r = \frac{1}{\delta} - 1$ or $\delta = \frac{1}{1+r}$. Similarly, for a nonshirker we get

$$\begin{aligned}
& rV_E^N = w - e + b[V_u - V_E^S] && (10.2.3) \\
\Rightarrow\;& V_E^N = \frac{w - e + bV_u}{r+b}. && (10.2.4)
\end{aligned}$$

The worker will choose not to shirk if and only if $V_E^N \geq V_E^S$. This is known as the *no-shirking condition (NSC)*, which, using (10.2.2) and (10.2.4), can be written as

$$w \geq rV_u + \frac{(r+b+q)}{q}e \equiv \hat{w}. \qquad (10.2.5)$$

Alternatively, the NSC also takes the form

$$q(V_E^S - V_u) \geq e. \qquad (10.2.6)$$

This highlights the basic implication of the NSC: unless there is a penalty associated with being unemployed, everyone will shirk. In other words, if an individual could immediately obtain employment after being fired, $V_E^S = V_u$, then the NSC could never be satisfied.

Equation (10.2.5) has several natural implications. If the firm pays a sufficiently high wage, then the workers will not shirk. The critical wage, $\hat{w}$, is higher

(a) The higher the required effort (e),

(b) The higher the expected utility associated with being unemployed (V_u),

(c) The lower the probability of being detected shirking (q),

(d) The higher the rate of interest (i.e., the relatively more weight is attached to the short-run gains from shirking (until one is caught) compared to the losses incurred when one is eventually caught),

[3]The original model is in continuous time, the mathematics of which is a bit involved. Hence, we do this in a discrete time setting. To keep notations similar, we assume that the workers get their wages and incur effort cost, if any, at the beginning of the next period.

(e) The higher the exogenous quit rate b (if one is going to have to leave the firm anyway, one might as well cheat on the firm).

10.2.2.3 Employers

There are M identical firms, $i = 1, ..., M$. Each firm has a production function $Q_i = f_i(L_i)$, generating an aggregate production function of $Q = F(L)$.[4] Here L_i is firm i's effective labour force; we assume a worker contributes one unit of effective labor if he does not shirk. Otherwise he contributes nothing. Therefore firms compete in offering wage packages, subject to the constraint that their workers choose not to shirk. It is assumed that $F^{'}(N) > e$, that is, full employment is efficient.

The monitoring technology (q) is exogenous. It is assumed that other factors (for example, exogenous noise or the absence of employee specific output measures) prevent monitoring of effort via observing output.

A firm's wage package consists of a wage, w, and a level of unemployment benefits, $\bar{w}$. Each firm finds it optimal to fire shirkers, since the only other punishment, a wage reduction, would simply induce the disciplined worker to shirk again.

Observe that all firms offer the smallest unemployment benefits allowed (say, by law). This follows directly from the NSC, equation (10.2.5). An individual firm has no incentive to set $\bar{w}$ any higher than necessary. An increase in $\bar{w}$ raises V_u and hence requires a higher w to meet the NSC. Therefore, increase in $\bar{w}$ cost the firm both directly (higher unemployment benefits) and indirectly (higher wages). Hence we can interpret $\bar{w}$ in what follows as the minimum legal level, which is offered consistently by all firms.

Having offered the minimum allowable $\bar{w}$, an individual firm pays wages sufficient to induce employee effort, that is, $w = \hat{w}$ to meet the NSC. The firm's labour demand is given by equating the marginal product of labor to the cost of hiring an additional employee. This cost consists of wages and future unemployment benefits. For $\bar{w} = 0$, the labour demand is given simply by $f^{'}(L_i) = \hat{w}$, with aggregate labor demand of $F^{'}(L) = \hat{w}$

10.2.2.4 Market Equilibrium

We now turn to the determination of the equilibrium wage and employment levels. Let us first indicate heuristically the factors which determine the equilibrium wage level.

If wages are very high, workers will value their jobs for two reasons: (a) the high wages themselves, and (b) the correspondingly low level of employment (due to low demand for labor at high wages) which implies long spells of unemployment in the event of losing one's job. In such a situation employers will find they can reduce wages without tempting workers to shirk.

---

[4]That is,

$$F(L) \equiv \max_{\{L_i\}} \sum f_i(L_i)$$

such that $\sum L_i = LE$. This assumes that in market equilibrium, labour is efficiently allocated.

Conversely, if the wage is quite low, workers will be tempted to shirk for two reasons: (a) low wages imply that working is only moderately preferred to unemployment, and (b) high employment levels (at low wages there is a large demand for labor) imply unemployment spells due to being fired will be brief. In such a situation firms will raise their wages to satisfy the NSC.

Equilibrium occurs when each firm, taking as given the wages and employment levels at other firms, finds it optimal to offer the going wage rather than a different wage. The key market variable which determines individual firm behavior is V_u, the expected utility of an unemployed worker. We turn now to the calculation of the equilibrium V_u. The asset equation for V_u, analogous to (10.2.1) and (10.2.3), is given by

$$rV_u = \bar{w} + a(V_E - V_u) \tag{10.2.7}$$

where a is the job acquisition rate and V_E is the expected utility of an employed worker (which equals V_E^N in equilibrium – why?). We can now solve (10.2.4) and (10.2.7) simultaneously for V_E and V_u to yield

$$rV_E = \frac{(w-e)(a+r)+\bar{w}b}{a+b+r} \tag{10.2.8}$$

$$rV_u = \frac{(w-e)a+\bar{w}(b+r)}{a+b+r}. \tag{10.2.9}$$

Substituting the expression for Vu (i.e., (10.2.9)) into the NSC (10.2.5) yields the aggregate NSC

$$w \geq \bar{w} + e + \frac{a+b+r}{q}e \tag{10.2.10}$$

Notice that the critical wage for nonshirking is greater: (a) the smaller the detection probability q; (b) the larger the effort e; (c) the higher the quit rate b; (d) the higher the interest rate r; (e) the higher the unemployment benefit ($\bar{w}$); and (f) the higher the flows out of unemployment a.

We commented above on the first four properties; the last two are also unsurprising. If the unemployment benefit is high, the expected utility of an unemployed individual is high, and therefore the punishment associated with being unemployed is low. To induce individuals not to shirk, a higher wage must be paid. As a, the probability of obtaining a job, increases the punishment associated with unemployment is low. Hence, we need larger wage to induce nonshirking.

The rate a itself can be related to more fundamental parameters of the model, in a steady-state equilibrium. In steady state the flow into the unemployment pool is bL where L is the aggregate employment. The flow out is $a(N-L)$ (at each period) where N is the total labor supply. These must be equal, so $bL = a(N-L)$, or

$$a = \frac{bL}{N - L} \tag{10.2.11}$$

Substituting for a into (10.2.10), the aggregate NSC, we have

$$\begin{aligned} w \geq & \bar{w} + e + \frac{e}{q}\left[\frac{bN}{N-L} + r\right] \\ = & \bar{w} + e + \frac{e}{q}\left[\frac{b}{u} + r\right] \equiv \hat{w} \end{aligned} \tag{10.2.12}$$

where $u = \dfrac{N-L}{N}$, the unemployment rate. This constraint, the aggregate NSC, is graphed in Figure 10.7.

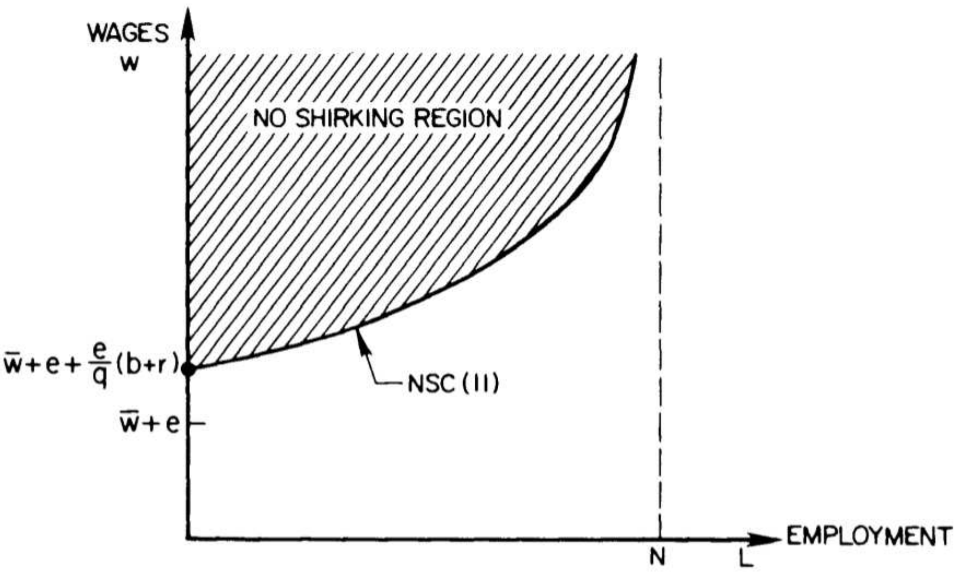


Figure 10.7: The Aggregate No-Shirking Condition

It is immediately evident that *no shirking is inconsistent with full employment.* If $L = N$, $a = +\infty$, so any shirking worker would immediately be rehired. Knowing this, workers will choose to shirk. The equilibrium wage and employment level are now easy to identify. Each (small) firm, taking the aggregate job acquisition rate a as given, finds that it must offer at least the wage $\hat{w}$. The firm's demand for labor then determines how many workers are hired at the wage. Equilibrium occurs where the aggregate demand for labor intersects the aggregate NSC. For $\bar{w} = 0$, equilibrium occurs when

$$F'(L) = e + \frac{e}{q}\left[\frac{bN}{N-L} + r\right]$$

The equilibrium is depicted in Figure 10.8. It is important to understand the forces which cause E to be an equilibrium. From the firm's point of view, there is no point in raising wages since workers are providing effort and the firm can get all the labor it wants at w^*. Lowering wages, on the other hand, would induce shirking and be a losing idea.

From the worker's point of view, *unemployment is involuntary*: Those without jobs would be happy to work at w^* or lower, but cannot make a credible promise not to shirk at such wages.

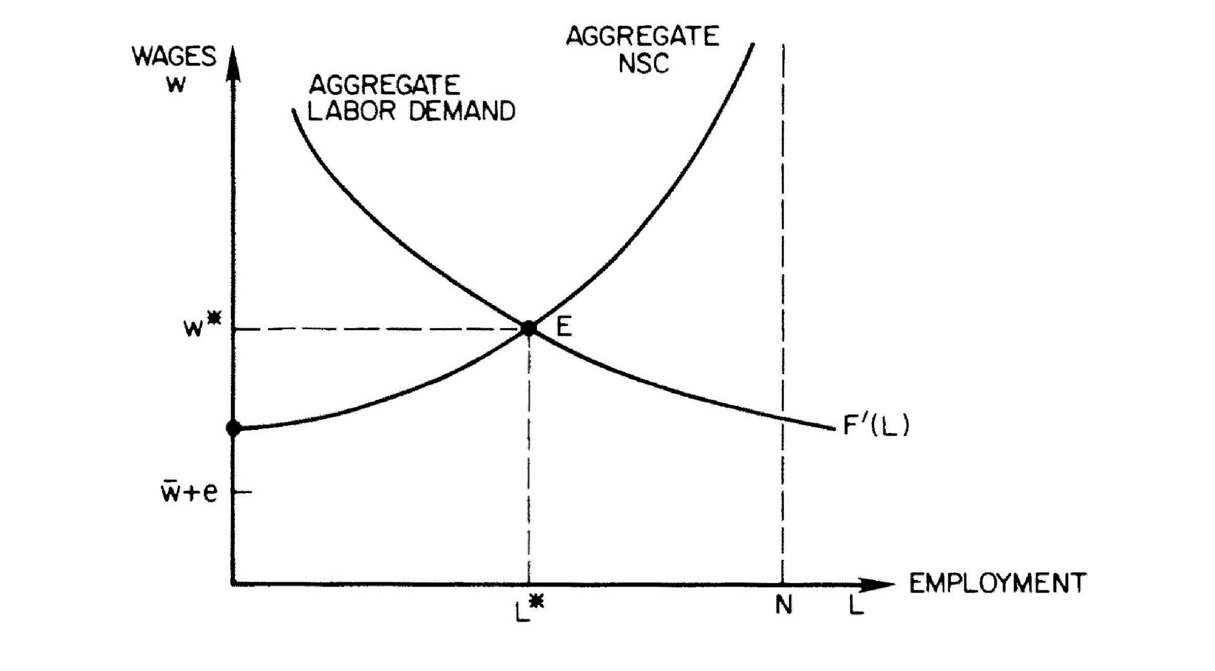


Figure 10.8: Equilibrium Unemployment

10.2.2.5 Comparative Statics

The effect of changing various parameters of the problem may easily be determined. As noted above, increasing the quit rate b, or decreasing the monitoring intensity q, decreases incentives to exert effort. Therefore, these changes require an increase in the wage necessary (at each level of employment) to induce individuals to work, that is, they shift the *NSC curve* upwards (see Figure 10.9). On the other hand, they leave the demand curve for labor unchanged, and hence the equilibrium level of unemployment and the equilibrium wage are both increased. Increases in unemployment benefits have the same impact on the *NSC curve*, but they also reduce labor demand as workers become more expensive, so they cause unemployment to rise for two reasons.

Inward shifts in the labor demand schedule create more unemployment. Due to the NSC, wages cannot fall enough to compensate for the decreased labour demand. The transition to the higher unemployment equilibrium will not be immediate: Wage decreases by individual firms will only become attractive as the unemployment pool grows. This provides an explanation of wage sluggishness.

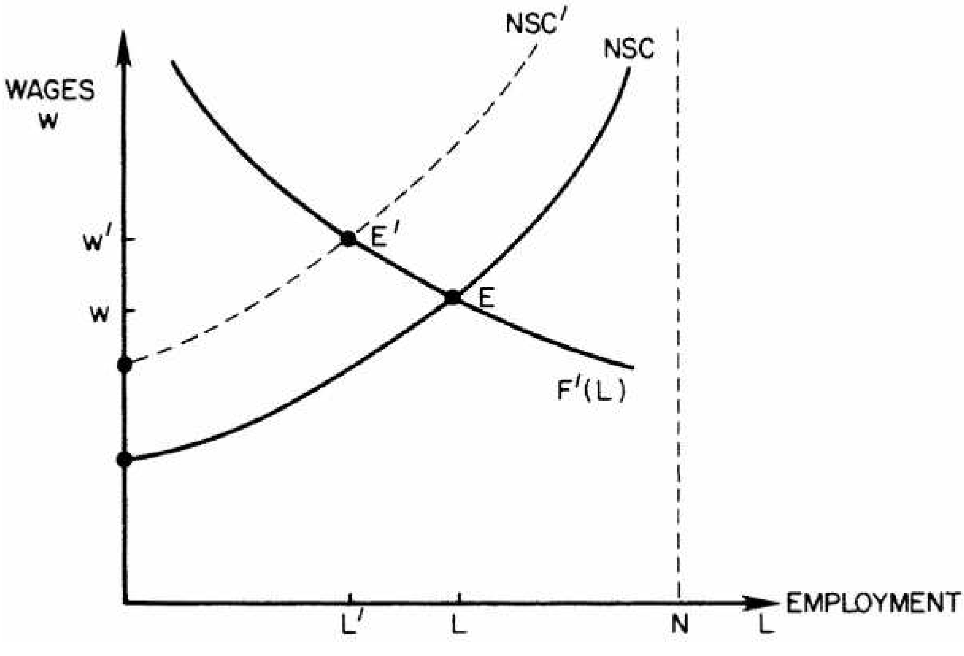


Figure 10.9: Comparative Statics

10.2.2.6 Welfare Analysis

The equilibrium level of unemployment is in general inefficient. Each firm tends to employ too few workers, since it sees the private cost of an additional worker as w (which must be greater than $\hat{w}$ which is, in turn, greater than e), while the social cost is only e, which is lower. On the other hand, when a firm hires one more worker, it fails to take account of the effect this has on V_u (by reducing the size of the unemployment pool). This effect, a negative externality imposed by one firm on others as it raises its level of employment, tends to lead to over-employment. In the model above, the former effect dominates, and the natural level of unemployment is too high.