A Theory of Progressive Lending*

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Abstract

We characterize Pareto efficient long term 'relational' lending contracts with one-sided lender commitment, to finance specific investments and smooth intertemporal consumption of a borrower who cannot commit to repaying loans. The borrower can save and accumulate assets, and has strictly concave preferences over consumption. For borrowers with initial wealth above a threshold, the first-best is sustainable with a stationary contract. For poorer agents, there is perpetual but shrinking underinvestment which disappears in the long run. Borrowing, investment and wealth grow and converge to the first-best threshold. Optimal allocations can be implemented by back-loaded 'progressive' lending: a sequence of one period loans of growing size. Increased borrower bargaining power raises short run consumption and investment, with no long term effects. We discuss extensions to incorporate random productivity shocks.

Keywords: Dynamic Contracts, Progressive Lending, Microfinance.

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1 Introduction

In this paper, we study Pareto efficient lending contracts between a lender that can commit, and a borrower that cannot commit to a long term relationship. The borrower accumulate assets via either of two ways: (a) saving at a constant interest rate equal to her intertemporal discount rate, and (b) investing in a strictly concave production technology. Moreover, the borrower (but not the lender) has a strictly concave utility over consumprion, i.e., prefers to smooth consumption over time. The borrower's outside option in the event of a loan default are increasing in wealth accumulated so far, and is given by present value utility from Ramsey-optimal saving and investment strategies under autarky. Productive investments are at least partly 'specific' in the sense that the investment technology in autarky generates a lower marginal return. In the absence of such specificity, it is well known (e.g., Bulow and Rogoff (1989), Rosenthal (1991)) that no lending contract exists that would enable the lender to break even and the borrower to be better off compared with autarky.

The model is well-suited to contexts where financial institutions (banks or microfinance institutions (MFIs)) lend to poor borrowers that lack collaterizable wealth, by building long-term lending relationships. It also applies to firms or employers that finance training of workers in specific skills via long-term employment relationships. In both contexts, borrowing agents are poor, seek to smooth consumption and finance investment to accumulate assets, and cannot commit to defaulting on loans or quitting. First-best allocations require first-best (stationary) investments and borrowers to borrow from their future wealth to smooth consumption, but are typically infeasible owing to the moral hazard associated with default. To overcome the moral hazard, lenders need to limit lending initially and 'backload' borrower incentives by offering larger loans in the future conditional on repayment of current loans. This reduces investment and thus limits the growth of borrower's wealth, besides preventing consumption smoothing. Hence by restricting backloading, consumption smoothing preferences of the agent reduce the scope of loans to generate higher asset growth and welfare. The aim of this paper is to study resulting short run and long run implications of these distortions.

Previous analyses of such contexts (Thomas and Worrall (1994), Albuquerque and Hopenhayn (2004)) have confined attention to the case where the borrower has linear utility and subject to limited liability. In the absence of preferences for intertemporal consumption smoothing, optimal contractsinvolve maximal backloading: the agent is not permitted to consume until investment distortions disappear. Such extreme backloading is no longer possible when the borrower has strictly concave utility, and optimal contracts need to trade off consumption smoothing and backloading. Ray (2002) considers a model of self-enforcing contracts with one-sided commitment and non-transferable utility, but in a repeated game context where the agent cannot save or invest. Hence consequences for investment and growth are not explored in his paper. The recent work of Thomas and Worrall (2017) studies a context with two agents with concave utility who contribute effort to a common joint output. Their setting differs from ours by incorporating lack of commitment of both parties, and not allowing either agent to save or invest.

We show that the lender can without loss of generality restrict attention to a sequence of one-period loans offered conditional on repayment of past loans and achievement of stipulated investment targets. Moreover, loan contracts are conditioned on a single state variable, a measure of net wealth of the borrower equal to value of current output, less debt repayments due. We obtain a recursive representation of Pareto efficient contracts maximizing welfare of an agent with arbitrary initial net wealth, subject to a minimum profit target for the lender, and no-default incentive constraints. This representation simplifies the analysis by enabling us to represent the incentive constraints in a tractable manner: the value of continuing the relationship the next period onwards should not fall below the outside option corresponding to the Ramsey autarkic value starting with an endowment equal to the current output, but with the inferior autarkic technology. We use this recursive representation to characterize the dynamics of the agent's investment, wealth and consumption, and how these are affected by changes in various parameters.

Our main results are the following. Agents with initial wealth above some threshold can achieve first-best welfare, in a steady state where investment is first-best and consumption perfectly smoothed. Hence we focus thereafter on poor agents, who start with a wealth below this threshold. We show that the optimal strategy can be implemented by progressive lending: a sequence of one-period contracts conditioned on current net wealth (gross output less debt due), with continuation predicated on loan repayment and achievement of productivity target. Loan amounts are strictly increasing in net wealth. Investment and consumption grow over time, though allocations are distorted at every date. Our main result is that these distortions eventually vanish: *the allocation converges to the steady state corresponding to the first-best wealth threshold, irrespective of initial conditions or consumption smoothing preferences.* Hence a debt trap never arises, irrespective of the extent of the principal's bargaining power, concavity of the borrower's payoff or the rate of time discount. The dynamic is qualitatively similar to that in a Ramsey model of autarkic self-financing, except that the steady state involves higher welfare, and convergence is faster (i.e., consumption, productivity and wealth are higher at every date).

Here is a sketch of the underlying argument.

(i) The recursive representation of the contracting problem implies optimal investment is a function of the agent's current net wealth. Owing to concavity of utility in current consumption, wealthier agents face a lower marginal cost of investing. Hence investment is non-decreasing in wealth. This implies that the sequence of wealths is monotonically increasing or decreasing over time. Therefore wealth must converge.

(ii) Since net wealth converges, so must consumption. Hence for large enough t, the agent's consumption must be smoothed nearly perfectly. This implies the investment distortion must also vanish, since the agent can always self-finance some extra investment. A first-best allocation must be attained in the limit, and the agent's limiting wealth must be at least w^*, the first-best threshold.

(iii) However a first-best allocation cannot be achieved in finite time, because this would result in a consumption distortion without a co-existing investment distortion.[1] Hence the agent's wealth must be strictly less than w^* at all dates. This is only possible if wealth is rising and converging to w^*.

(iv) The argument for optimality of progressive lending (i.e., loan sizes are increasing in net wealth) is somewhat more involved, and is based on showing that consumption grows faster on the equilibrium path than in the counter-factual event of default. Intuitively, this is because the technology available to the agent on the equilibrium path has a higher rate of return than the autarkic technology. Since the incentive constraint binds, the present value of consumption is the same on and off the equilibrium path. Therefore current consumption must be lower on the equilibrium path. This implies a higher marginal welfare impact of increasing wealth on the equilibrium path, i.e. wealthier borrowers can credibly commit to repaying larger loans.

Comparative dynamics with respect to increased 'aid' (or more generally, a decrease in the lender's profit target, or the agent's bargaining power) yields short-run increases in investment and consump-

[1] Specifically, if the first-best could be attained at some finite date T, consumption would be smooth after T, and strictly higher than at $T-1$. Moreover output at T would be first-best, which requires first-best investment at $T-1$. In other words, the distribution of consumption between T and $T-1$ would be distorted, while investment is not. This cannot be optimal: the agent could reduce investment slightly at $T-1$ so as to reduce the consumption distortion, while the reduction in output at T would have a zero first order welfare effect.

tion, while leaving the long-run allocation unaffected. Hence the benefits of aid are entirely front-loaded, with no long-term consequences. In the context of intra-firm skill training, the model predicts lifetime wage and productivity age-profiles shift upward and become flatter when workers' bargaining power improves (as a result of unions or competition among firms for workers).[2]

The last section of the paper extends the model to incorporate productivity shocks. A recursive representation continues to apply, and investments increase in net wealth. Moreover, conditional on stationary (or nondecreasing) productivity shocks, the agent's net wealth, investment and output increase over time, with underinvestment disappearing once net wealth crosses some threshold. Of course in this setting wealth could also contract, owing to the possibility of declining productivity shocks. If wealth falls from one period to the next owing to an adverse shock, it continues to fall thereafter if the shock persists or becomes worse over time. Hence the wealth dynamic continues to be qualitatively similar to the neoclassical growth model.[3]

Our model therefore provides a rationale for progressive lending both in a positive and normative sense. Loan sizes increase, conditional on past repayment, and loans are repaid on the equilibrium path, broadly consistent with observed practice of MFIs.[4] These strategies provide repayment incentives efficiently, ensure that the lender's profit targets are met, and enable poor borrowers to escape poverty and accumulate wealth (conditional on absence of adverse shocks).[5]

Section 2 discusses relation to existing literature in more detail. Sections 3 and 4 provide analyses for the deterministic and uncertainty contexts respectively. Proofs are collected in the Appendix.

[2]The empirical evidence on this appears to be somewhat mixed, as discussed in Acemoglu and Pischke (1998).

[3]We conjecture that results concerning ergodicity of wealth dynamics also continue to hold, but do not pursue this further.

[4]Table 1 shows that MFIs experience remarkably low default rates, while Table 2 provides evidence of use of progressive lending strategies by many large MFIs in Bangladesh, India and Vietnam.

[5]This is in an 'ideal' setting with rational agents with dynamically consistent preferences, convex technologies, and lender commitment to exclusive long-term credit contracts.Exclusivity clauses prevent borrowers from switching to a new lender after defaulting. These can arise owing to lender's coordinating on strategies, or borrowers being institutionally prevented from borrowing simultaneously from multiple lenders. See Pauly (1974), Arnott and Stiglitz (1986), Bizer and deMarzo (1992) or Kahn and Mookherjee (1998). To the extent that the observed anti-poverty impact of microcredit so far have been less successful than the model predicts (e.g., see American Economic Journal: Applied Economics (2015)), our results suggest that the problem does not stem from any inherent defect in progressive lending policies *per se*. Instead they must lie in violation of one or more of the other ideal assumptions.

2 Related Literature

As mentioned in the Introduction, Thomas and Worrall (1994), Albuquerque and Hopenhayn (2004) study the case where the agent has no preferences for consumption smoothing, and the extent of back-loading is limited by a limited liability constraint. In an extension of their main model, Thomas and Worrall (1994) show their main results extend when the utility function is 'not too' concave. With concave utility, the analysis is considerably more complicated owing to the need to characterize consumption dynamics via an Euler equation respecting incentive constraints. Our model allows us to study the effect of consumption smoothing preferences on the dynamics of investment and wealth. Our analysis and results apply irrespective of the concavity of the utility function, and we show that it does not affect long run allocations. Owing to the maximal backloading in the absence of smoothing preferences, the effects of increased aid are different: the benefits would accrue in later periods and the agent would continue to consume nothing initially. In contrast with concave utility, the optimal contract will strike a balance between front-loading and back-loading, and the benefits of greater aid will appear only at the beginning.

Ray (2002) considers a general model of constrained Pareto efficient self-enforcing contracts in the context of a repeated game between a principal and agent with limited transferability of utility, where the agent cannot save nor commit. He shows that all such contracts back-load in the sense the allocation of surplus tends progressively to the agent's favor at later dates, converging to the one that maximizes the agent's continuation payoff, which may or may not involve a distortion. Our model imposes more structure on preferences and technology, but incorporates investment and wealth accumulation, and shows that distortions disappear eventually. We also obtain a backloading result analogous to his: the optimal allocation involves an initial adjustment of the agent's initial wealth with a lump-sum transfer to the principal, followed by a contract that maximizes the agent's continuation utility subject to a breakeven constraint for the principal.

Thomas and Worrall (2017) study optimal relational contracts between two agents neither of whom can commit, both contribute effort to a common joint output, and cannot save or invest. They consider both the case where the agents have strictly concave utility, and where they have linear utility and subject to limited liability. In the former case, their results turn out similar to ours: over-investment never occurs; and convergence to the first-best is monotone if the first-best is sustainable.

A number of papers on dynamic lending in microfinance focus on unobserved borrower heterogeneity, and the possible role of progressive lending in screening borrowers. In all of these models, defaults necessarily occur on the equilibrium path. This limits their relevance to the microfinance setting where repayment rates are near 100%. For instance, Ghosh and Ray (2016) show how progressive lending can help screen out bad borrowers who always default from good borrowers, by providing small initial loans followed by larger ones after the bad borrowers have been eliminated. Egli (2004) shows that progressive lending may fail to identify a "bad" type, since a bad borrower may camouflage herself as a "good" borrower (who always repays) in order to get a higher amount of loan later on which she defaults with certainty. Shapiro (2015) examines a framework with uncertainty over borrowers' discount rates. He shows that even in the efficient equilibrium almost all the borrowers default. An earlier paper with the same feature is Aghion and Morduch (2000), where in a two period context the borrower repays the first period loan to get a higher amount of loan in the second period on which she subsequently defaults. In contrast to these papers, we provide a theory of progressive lending without any equilibrium default and without any unobserved heterogeneity.

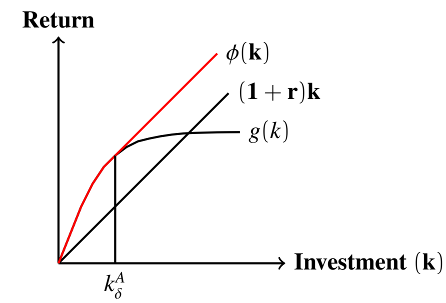
Dasgupta and Roy Chowdhury (2018) provide an alternative explanation of progressive lending in a framework with a nonconvexity, where the borrower (with linear utility) has an opportunity to graduate to a higher occupation or productive activity which requires a minimum investment. The MFI provides a savings as well as borrowing opportunity which enables the borrower to accumulate wealth and graduate at an endogenous finite date. Over time, savings with the MFI increase which the borrower forfeits in case of default. The endogenous growth of collateral permits the lender to extend larger loans which are repaid. In a context with a similar nonconvexity, Liu and Roth (2020) present a model of a debt poverty trap that arises when the lender is profit-maximizing and cannot commit to long term contracts. Finally Mookherjee and Ray (2002) present a model where poverty traps can arise without any technological nonconvexity, with a profit-maximizing lender with limited commitment.

3 Model with Deterministic Technology

Consider an agent with endowment e, in an infinite horizon discrete time framework. Her current payoff is given by $u(c)$ where c denotes consumption. $u(\cdot)$ is time-stationary, twice differentiable, strictly increasing, strictly concave and satisfying Inada conditions. The agent's objective is to maximize the present discounted value of her lifetime utility: $\sum_{t=0}^{\infty} \delta^t u(c_t)$; where $\delta \in (0, 1)$ is the discount factor.

3.1 Autarky

The agent always has access to (i) a neoclassical production technology $g(\cdot)$ which is strictly increasing, strictly concave and satisfies the Inada conditions: $g'(0) = \infty$ and $g'(\infty) = 0$ and (ii) a linear savings opportunity at a constant rate of return $r = \frac{1}{\delta} - 1$. Together, these imply that the agent has access to a transformation possibility of resources from any date t to $t + 1$ at a rate of return bounded below by $r = \frac{1}{\delta} - 1$. This possibility is represented by the function $\phi(k)$ which provides total resources available at the next date if the agent invests a total of k, shown in Figure 1. The budget constraint at t is then $c_t + k_{t+1} \leq \phi(k_t)$, where c_t, k_t denote capital stock at t.



$$\phi(k) = \begin{cases} g(k) & if \;\; k \leq k^A_\delta \\ g(k^A_\delta) + (1+r)(k - k^A_\delta) & \text{otherwise.} \end{cases}$$

Figure 1: Autarky Technology

This formulation assumes that capital depreciates entirely after one period. As is well known, it is easy to extend the model to incorporate less than 100% depreciation, by redefining the production function. Suppose $k_{t+1} = (1-d)k_t + i_t$, where d is the proportion of capital stock that depreciates in one period, i_t denotes investment, k_t denotes capital stock, and output $y_t = y(k_t)$ at date t, where $y(.)$ is a strictly increasing, concave function satisfying $y(0) = 0$ and $y'(0) = \infty$.The agent's budget constraint at t is then $c_t + i_t \leq y((1-d)k_{t-1} + i_{t-1})$, which can be rewritten as $c_t + k_{t+1} \leq \phi(k_t) \equiv y(k_t) + (1-d)k_t$, with $\phi(0) = 0, \phi'(0) = \infty$. In the remainder of the paper, we will therefore simplify notation by assuming full depreciation.

Letting k^A_δ denote the solution to $\delta g'(k) = 1$, in autarky the problem of the agent with endowment

$e > 0$ is to

$$\begin{aligned}&\underset{\{k_{\tau+1}\}_{\tau=0}^{\infty}}{\text{Maximize}}\ u\Big(e-k_1\Big)+\sum_{\tau=1}^{\infty}\delta^{\tau-t}u\Big(\phi(k_\tau)-k_{\tau+1}\Big)\\&\text{Subject to:}\\&k_1\leq e \text{ and } \quad k_{\tau+1}\leq\phi(k_\tau),\forall\tau\geq 1.\end{aligned}$$

We denote the solution to this standard Ramsey optimal growth problem by $V^A(e)$. It is well known that V^A is differentiable, with $V^{A'}(e) = u'(e - k_A(e))$, where $k_A(e)$ is the optimal investment rule which is nondecreasing in e.

3.2 The Lender

The lender provides the agent with access to a more productive technology $z(k)$, as well as to loans. The cost of capital of the lender is r, equal to the agent's rate of return on savings. The production function $z(.)$ dominates $g(.)$ in terms of both absolute and marginal returns to investment: $z(k) > g(k)$ and $z'(k) > g'(k)$ for all k. This can represent a higher TFP, or a higher price at which the output can be sold. Alternatively, it could be a different production opportunity subject to decreasing returns, which enables the agent to raise returns by allocating investment between the autarkic technology and the new one.[6] Let k_δ denote the first-best investment, which solves $\delta y'(k) = 1$. And let $y(k)$ denote the return on investment k which is optimally allocated between production and savings, so that

$$y(k)=\begin{cases}z(k) & if \quad k\leq k_\delta\\ z(k_\delta)+(1+r)(k-k_\delta) & \text{otherwise.}\end{cases}$$

3.2.1 Credit Contracts

The agent is subject to *ex post* moral hazard and cannot commit to repaying loans. The lender on the other hand can commit to a long-term contract providing access to the improved technology, stipulated investments and financial transfers to the agent, as a function of past history which includes past investments, loans and repayments (all of which are verifiable).

[6]If the production function in the new opportunity is represented by a strictly concave twice-differentiable function $f(k)$ satisfying Inada conditions, the agent will optimally allocate a total investment of k into $x(k)$ in the new technology and $k - x(k)$ in the old, such that $f'(x(k)) = g'(k - x(k))$. This results in total production $z(k) = f(x(k)) + g(k - x(k))$ which dominates the old one in the sense described.

Standard arguments imply that attention can be restricted without loss of generality to contracts where all loans are repaid on the equilibrium path, and any default is followed by suspension of technology and loan access at all future dates. Hence we can focus attention on incentive compatible contracts. A contract is a sequence $p \equiv \{p_0, p_1, ...\}$ and $k \equiv \{k_0, k_1, ...\}$ of stipulated net transfers and investments at each date $t = 0, 1, ..$, conditional on absence of any past defaults. When $p_t < 0$, the borrower effectively obtains a loan, while $p_t > 0$ denotes payments made by the borrower.

Let e denote the initial endowment of the borrower, and π an arbitrary profit target for the lender. A Pareto optimal contract then solves the following problem.

$$\max_{\left\langle \{p_t\}_{t=0}^{\infty}, \{k_t\}_{t=0}^{\infty} \right\rangle} V_0 \equiv [u(e + p_0 - k_0) + \sum_{t=1}^{\infty} \delta^t u(y(k_{t-1}) + p_t - k_t)] \tag{1}$$

subject to:

Lender's Profit Constraint

$$\text{LPC}: \quad p_0 + \sum_{t=1}^{\infty} \delta^t p_t \leq -\pi$$

Incentive Compatibility Constraints

$$\text{IC}_t: \quad V_t \equiv \sum_{\tau=t}^{\infty} \delta^{\tau-t} u(y(k_{\tau-1}) + p_\tau - k_\tau) \geq V^A(y(k_{t-1})), \quad \forall t \geq 1.$$

Borrower Participation Constraint

$$\text{BPC}: \quad V_0 \geq V^A(e).$$

We do not include non-negativity constraints on the agent's consumption owing to Inada conditions. The borrower has the option to default at any date $t \geq 1$; hence the incentive compatibility constraint requires that continuation payoff does not fall below the autarkic payoff corresponding to an endowment equal to the current output $y(k_{t-1})$. The participation constraint requires an analogous condition at $t = 0$, where the borrower's outside option corresponds to the autarkic payoff starting with an endowment e.

We start with a preliminary observation which enables the problem to be simplified.

Lemma 1. *(a) There exists an upper bound $\bar{\pi}(e) \in (0, \infty)$ to the profit that can be earned by the lender in any feasible contract.*

(b) Given any feasible profit target $\pi \leq \bar{\pi}(e)$, the optimal contract solves the following competitive

equilibrium (CE) problem with initial endowment $w \equiv e - \pi$:

$$V(w) \equiv \max_{\left\langle \{p_t\}_{t=0}^{\infty}, \{k_t\}_{t=0}^{\infty} \right\rangle} [u(w + p_0 - k_0) + \sum_{t=1}^{\infty} \delta^t u(y(k_{t-1}) + p_t - k_t)] \tag{2}$$

subject to:

Lender's Breakeven Constraint

$$LBC: \quad p_0 + \sum_{t=1}^{\infty} \delta^t p_t \leq 0$$

Incentive Compatibility Constraints

$$IC_t: \quad V_t \equiv \sum_{\tau=t}^{\infty} \delta^{\tau-t} u(y(k_{\tau-1}) + p_\tau - k_\tau) \geq V^A(y(k_{t-1})), \quad \forall t \geq 1. \tag{3}$$

(a) states there is a positive upper bound $\bar{\pi}(e)$ to the profit that the lender can earn by contracting with a borrower with initial endowment $e > 0$: any higher profit would violate the borrower's participation constraint. Any lower profit than $\bar{\pi}(e)$ allows the existence of a feasible contract. The upper bound is positive since there is always a feasible contract if $\pi = 0$: access to the more profitable technology allows the borrower to attain a higher payoff compared to autarky in the absence of any transfers (i.e., if $p_t = 0$ for all t). Hence the borrower would be willing to pay a positive fee for such access, even if it is not bundled with any loans. Bundling with loans will further increase the scope for earning profits.

Part (a) therefore implies we can focus attention on the case where $\pi \leq \bar{\pi}(e)$. The borrower participation constraint can then be dropped. Part (b) says moreover that the optimal contracting problem can be simplified as follows. First modify the borrowers initial endowment from e to $w = e - \pi$, i.e., via a lump-sum transfer of π from the borrower to lender. Then the lender provides transfers p_t at successive dates to the borrower, subject to a break-even constraint, besides the repayment incentive constraints. It therefore suffices to focus on the problem in (b), which corresponds to the CE contract maximizing payoff of a borrower with initial endowment $w = e - \pi$.

In the remainder of the paper, we therefore study the CE problem (b). We shall refer to $w = e - \pi$ as the borrower's initial wealth and treat this as a parameter.

It is evident that the borrower's CE payoff $V(w)$ is strictly increasing in her initial wealth, since this permits the borrower to consume more at $t = 0$ without disturbing any of the incentive constraints (which pertain to $t \geq 1$). Moreover, this payoff must strictly exceed her outside option $V^A(w)$ corresponding to an initial endowment of w, because it is always feasible for a lender to break even by providing the borrower with access to the technology $y(.)$ unaccompanied by any financial transfers. Moreover, the optimal contract can be implemented by a sequence of one period loans at the competitive interest rate r, which satisfy a 'no Ponzi' scheme condition:

Lemma 2. *(a) $V(.)$ is strictly increasing, and $V(w) > V^A(w)$ for all w.*

(b) The optimal contract can be implemented by a sequence of one period loan contracts $l_t, t = 1, 2, ..$ charging the interest rate r, which are always repaid on the equilibrium path, where

$$l_0 = p_0, l_t = p_t + (1+r)l_{t-1}, \forall t \tag{4}$$

and

$$\lim_{T\to\infty} \delta^T l_T \leq 0. \tag{5}$$

(b) follows from observing that the payment sequence p_t is equivalently represented by the one-period loan sequence l_t defined by (4), so $p_t = l_t - \frac{l_{t-1}}{\delta}$. We can interpret the transfer p_t at any date as the composition of a fresh loan l_t that is offered, after the agent has repaid the previous loan. Observe that $\sum_{t=0}^{T} \delta^t p_t = \delta^T l_T$. Hence the break-even constraint (LBC) reduces to (5).

3.3 First-best Contracts

Consider the optimal contract when all the incentive constraints are dropped. It involves full consumption smoothing (via choice of transfers p_t) and efficient investment $k_t = k_\delta$. The constant consumption $c^*(w)$ is obtained by the requirement that the resulting present value of consumption $\frac{c^*(w)}{1-\delta}$ equals the present value of endowment/output minus investment: $w - k_\delta + \dfrac{\delta}{1-\delta}[y(k_\delta) - k_\delta]$, so

$$c^*(w) = (1-\delta)w + \delta y(k_\delta) - k_\delta. \tag{6}$$

The borrower then attains welfare $V^*(w) = \dfrac{u(c^*(w))}{1-\delta}$.

As the first best contract is stationary, all the incentive constraints collapse to a single constraint $c^*(w) \geq (1-\delta)y(k_\delta) + \delta g(k^A_\delta) - k^A_\delta$.[7] The first-best is incentive compatible if and only if the borrower's wealth exceeds the following threshold:

$$w^* \equiv y(k_\delta) - \frac{(\delta y(k_\delta) - k_\delta) - (\delta g(k^A_\delta) - k^A_\delta)}{1-\delta} \tag{7}$$

[7] If the borrower were to default, she would enter autarky with an initial wealth of $y(k_\delta)$. Since this exceeds $g(k^A_\delta)$ the autarkic steady state output, the agent would smooth consumption perfectly, and attain a per period consumption of $c^A = (1-\delta)y(k_\delta) + \delta g(k^A_\delta) - k^A_\delta$.

Since the $y(.)$ technology dominates $g(.)$, it follows that the threshold w^* is smaller than the first-best output $y(k_\delta)$. An agent with wealth in the interval $[w^*, y(k_\delta))$ obtains a loan at the first date of $\delta[y(k_\delta) - w]$ and then repays $(1 - \delta)[y(k_\delta) - w]$ at every subsequent date. These repayments are motivated by the lender threatening a defaulter with loss of access to the more productive technology, which allows the borrower to earn additional surplus $(\delta y(k_\delta) - k_\delta) - (\delta g(k_\delta^A) - k_\delta^A)$ at every date in the future. Observe that $\delta^T l_T = \delta^{T+2}[y(k_\delta) - w]$ which converges to 0 as $T \to \infty$, so the 'no Ponzi scheme' condition (5) is satisfied. This is sustained by a stationary sequence of one-period loans of size $\delta[y(k_\delta) - w]$ at every date. If w falls below w^* the loan is too large, resulting in a repayment obligation that would motivate the borrower to default.

For agents starting with wealth above $y(k_\delta)$, there is no need to borrow to achieve the first-best allocation: they invest k_δ in production, and supplement this by saving $\delta[w - y(k_\delta)]$ at every date.

Note that the incentive problem arises only for intermediate ranges of the discount factor. If δ approaches 1, the first best can be sustained for any initial w, as the threshold w^* goes to minus infinity. While if δ approaches zero, the threshold approaches zero (as in this case, the efficient investment approaches zero), and the demand for loans vanishes.

It follows the first-best contract is incentive compatible if and only if the borrower is wealthy enough to start with: $w \geq w^*$.

3.4 Second-best Contracts for Poor Borrowers

Now we focus on poor borrowers with $w < w^*$ and characterize the features of the optimal contract. Let $c_t = y(k_t) + p_t - k_t$ denote the agent's consumption at date $t \geq 0$.

Lemma 3. *$c_t \geq c_{t-1}$ for all t.*

Proof. Suppose otherwise, and $c_t < c_{t-1}$ for some t. Lower p_{t-1} slightly, and raise p_t correspondingly to keep $p_{t-1} + \delta p_t$ unchanged. This smooths consumption, raising V_l for every $l \leq t$, while leaving it unchanged for every $l > t$. Hence LBC and all incentive constraints are preserved, while raising borrower welfare. ∎

To make further progress we use Lemma 1 to obtain a recursive formulation of the problem in terms of one-period loans. We study the 'relaxed' problem where the asymptotic breakeven constraint (5) is ignored. This relaxed problem can be stated as

$$\underset{\langle\{l_t\}_{t=0}^{\infty},\{k_{t+1}\}_{t=0}^{\infty}}{\text{Maximize}}\left[u(w+l_0-k_0)+\sum_{t=1}^{\infty}\delta^t u\Big(y(k_{t-1})-\frac{l_{t-1}}{\delta}+l_t-k_t\Big)\right]$$

subject to

$$\text{IC: } V_t \equiv \sum_{\tau=t}^{\infty}\delta^{\tau-t}u\Big(y(k_{\tau-1})-\frac{l_{\tau-1}}{\delta}+l_\tau-k_\tau\Big) \geq V^A\Big(y(k_t)\Big) \quad \forall t \geq 1. \tag{8}$$

We will show later that the solution to this relaxed problem will end up automatically satisfying the breakeven constraint (5). Hence the optimal contract can be characterized by the solution to the relaxed problem. What makes this problem tractable is that the relaxed problem admits the following convenient recursive representation.

Observe that starting from any date t, the effect of past history is summarized in the single state variable $w_t \equiv y(k_{t-1})-\frac{l_{t-1}}{\delta}$, the borrower's net wealth which is the value of current output less inherited debt. So the contracting problem can be restated as follows.

Lemma 4. *The maximum attainable welfare $V(w)$ for a borrower with initial wealth w must satisfy*

$$V(w)=\max_{l,k}[u(w+l-k)+\delta V(y(k)-\frac{l}{\delta})] \quad \textit{subject to:} \quad V(y(k)-\frac{l}{\delta}) \geq V^A\Big(y(k)\Big) \tag{9}$$

Denote target wealth by $\Omega(l,k) \equiv y(k)-\frac{l}{\delta}$. Then (9) can be restated as:

$$V(w)=\max_{l,k}[u(w+l-k)+\delta V(\Omega(l,k)] \quad \text{subject to:} \quad V(\Omega(l,k)) \geq V^A(y(k)) \tag{10}$$

This problem can be broken into two stages. At the first stage, given any 'target' wealth Ω for the next date, select (l,k) to minimize the net investment cost, i.e, the sacrifice of current consumption $k-l$, subject to the incentive constraint $V(\Omega) \geq V^A(y(k))$. Let the resulting minimized cost be denoted by $C(\Omega)$. Formally,

$$C(\Omega)=\min_{l,k}(k-l) \quad \text{subject to:} \quad y(k)-\frac{l}{\delta}=\Omega \quad \text{and} \quad V(\Omega) \geq V^A(y(k)). \tag{11}$$

Then at the second stage, select the optimal target wealth $\Omega(w)$ for the next date, given current wealth w. We summarize this as follows.

Lemma 5. *The maximum attainable welfare $V(w)$ for a borrower with initial wealth w must satisfy*

$$V(w)=\max_{\Omega}[u(w-C(\Omega))+\delta V(\Omega)] \tag{12}$$

Let us start with the first stage cost minimization problem. Given target wealth Ω and capital choice k, the associated current loan must be $l(\Omega, k) = \delta y(k) - \delta\Omega$. Hence we can simplify (11) and reduce it to choice of investment alone as follows:

$$C(\Omega) = \delta\Omega + \min_k (k - \delta y(k)) \quad \text{subject to:} \quad V(\Omega) \geq V^A(y(k)) \tag{13}$$

So when $\Omega \geq w^*$, the agent invests k_δ from the very first period and hence $C(\Omega)$ in that case becomes $\delta\Omega - [\delta y(k_\delta) - k_\delta]$. While if $\Omega < w^*$, the incentive constraint binds and in particular

$$V(\Omega) = V^A\Big(y(k)\Big) \tag{14}$$

so the resulting investment size is $k(\Omega) = y^{-1}\Big(V^{A^{-1}}(V(\Omega))\Big)$. Since $V(\Omega) < V(w^*) = V^A\Big(y(k_\delta)\Big)$, and V^A, V and y are increasing, hence when $\Omega < w^*$ there will be underinvestment: $k(\Omega) < k_\delta$. We summarize this in the following lemma.

Lemma 6. *For target wealths Ω smaller than w^*, investment $k(\Omega)$ is smaller than the efficient level k_δ, and equal to the efficient level otherwise.*

So given target wealth Ω, optimal investment is uniquely determined:

$$k(\Omega) \equiv \begin{cases} k_\delta & \text{if } \Omega \geq w^* \\ y^{-1}\Big(V^{A^{-1}}(V(\Omega))\Big) & \text{otherwise} \end{cases}$$

and the cost function is

$$C(\Omega) = \delta\Omega + k(\Omega) - \delta y(k(\Omega)) \tag{15}$$

which is continuous and strictly increasing. Clearly the marginal cost of target wealth is δ for wealthy borrowers ($w > w^*$) and larger than δ for poor borrowers.

The second-stage problem involves choosing the target wealth Ω. Observe first that the set of attainable target wealths is bounded above by $\frac{w}{\delta}$, since the marginal cost of target wealth is bounded below by δ, and current consumption must be non-negative owing to the Inada conditions. It is also bounded below, e.g., by the incentive constraint which requires $V(\Omega) \geq V^A(y(0))$. Hence there always exists an optimal target wealth.[8] The optimal target wealth however may be non-unique. In what follows we consider any policy function which is a measurable selection from the optimal target wealth correspondence.

[8]This relies on the upper semi-continuity of the value function, which follows from a standard recursive argument.

Since V is strictly increasing, a higher target wealth is always valuable. The borrower must trade off a higher target wealth against the current cost. The concavity of u implies wealthier borrowers incur a lower marginal cost (in terms of sacrifice of utility from current consumption) of achieving higher future wealth. Hence those currently wealthier will remain wealthier in future.

Lemma 7. *$\Omega(w)$ is nondecreasing in w.*

One useful consequence of this result is the following.

Lemma 8. *The value function $V(w)$ is differentiable almost everywhere, with derivative equal to $u'(c(w))$, where $c(w) \equiv w - k(w) + l(w)$ denotes the agent's consumption policy. More generally, at every w, the right hand derivative of V is bounded below by $u'(c(w))$.*

We continue to focus on the case where initial wealth is below the threshold w^*. We are now ready to present our first main result, showing the absence of any poverty trap: the wealth of every poor agent will converge to the first-best threshold w^*.

Proposition 1. *If $w < w^*$, the sequence of net wealths w_t is strictly increasing, strictly smaller than w^* at every t, and converges to the first best threshold w^* as $T \to \infty$. The corresponding investment sequence k_t is nondecreasing and converges to k_δ, and consumption c_t is nondecreasing and converging to $c^*(w^*)$.*

The underlying intuition is the following. The sequence of net wealths is monotone, hence must converge. This implies that near the limit, consumption is almost perfectly stationary, i.e., the consumption distortion vanishes asymptotically. Then there cannot be any underinvestment in the limit. Otherwise it is feasible for the agent to increase welfare by raising investment slightly and finance it with a combination of altered borrowing and self-financing which preserves the incentive constraint. A similar logic (but reversed) ensures that convergence cannot be achieved in finite time (because that would imply absence of a production distortion and presence of a consumption distortion at the previous date).

Observe also that the no Ponzi scheme condition (5) holds for the optimal contract for any poor agent starting below w^*, because the wealth and behavior of such agents eventually are arbitrarily close to those of someone who starts with w^*. Hence as $T \to \infty$, $\delta^T l_T$ converges to $l(w^*) lim_{T\to\infty}\delta^T = 0$, and LBC is automatically satisfied at the solution to the relaxed problem in which it was dropped.

Our next main result is that the optimal strategy involves progressive lending: when $w < w^*$ optimum loan size increases over time.

Proposition 2. *Starting with any $w < w^*$, the borrower obtains a loan $l(w)$ which is strictly positive and strictly increasing in w.*

The reasoning is based on observing that the optimal loan size is characterized by the binding incentive constraint: $V(y(k(w)) - \frac{l(w)}{\delta}) = V_A(y(k(w))$ for all $w < w^*$. The loan size $l(w)$ is rising in w if $V'(\Omega(w)) = u'(c(\Omega(w))$ exceeds $V_A'(y(k(w)) = u'(c_A(y(k(w))))$, where $c_A(w')$ denotes the optimal consumption of an agent in autarky when starting with wealth w'. In other words, consumption on the equilibrium path lies below consumption on the corresponding autarkic outside option which generates equal welfare. This in turn holds because the borrower has access to a more productive technology on the equilibrium path, implying a faster rate of consumption growth. Since the present value of consumption is the same on and off the equilbrium path, current consumption on the equilibrium path must be lower.

4 Extension to Uncertain Productivity Shocks

In autarky, the output of the agent is now $g(k; s)$, where s is an i.i.d. shock with a CDF $J(.)$ over a finite support $[\underline{s}, \bar{s}]$, and g is twice differentiable, satisfying $g_k > 0 > g_{kk}, g_s > 0, g_{ks} \geq 0$ besides Inada conditions. This includes both the case of additive ($g(k; s) = \tilde{g}(k) + s$) and multiplicative shocks ($g(k; s) = s\tilde{g}(k)$). In the presence of the lender, the agent has a production function $y(k; s)$ with higher output and marginal product of capital than the autarkic technology at any $(k; s)$ and satisfying all other analogous properties of $g()$ mentioned above.

We assume that at any date t, the shock is observed *before* investment decisions are made. This is analogous to the model of Albuquerque and Hopenhayn (2004)). If the shock is observed after investments are made, it can be verified that all the results continue to apply if the shocks are additive.[9]

[9] The main complication arises from the need to keep track of a single dimensional measure of wealth the value of which can be targeted while making current investment decisions. If the current shock is known, the borrower and lender can predict the former's output that will result next period. If the shock is unknown, the borrower's output cannot be predicted with certainty. However, in this case if the shock is additive, the mean output can be predicted which serves to pin down the distribution of output as well as wealth next period by their corresponding certainty equivalents (CE). That is, with production function $\tilde{y}(k) + s$, the wealth next period from choice of (k, l) today will be $\tilde{w} + s$ where $\tilde{w} = \tilde{y}(k) - \frac{l}{\delta}$. Hence future CE wealth $(\tilde{w} + Es)$ is determined by current decisions, and a higher CE wealth results in a wealth distribution that is 'larger' in the sense of first order stochastic dominance. Hence we can extend the theory to this case where we replace target wealths by target CE wealths.

4.1 Autarky

The autarkic value function is for a state with current wealth e which is the value of output realized from investment at the previous date, and shock s applying to production at the current date:

$$V_A(e,s) = \max_{0 \le k \le w} [u(e-k) + \delta E_{s'}\{V_A(g(k;s), s')\}] \tag{16}$$

Standard arguments imply that optimal investment $k_A(e,s)$ is non-decreasing in both arguments. This implies that wealth follows a monotone Markov process (Hopenhayn and Prescott (1992)): the distribution of wealth at the next date conditional on current wealth w first order stochastically dominates that conditional on a lower current wealth $w' < w$. Note that unlike Hopenhayn and Prescott (1992) we have not imposed any exogenous upper bound on capital stocks, so the state space is not compact and their results concerning invariant or limiting distributions do not necessarily apply.

4.2 Access to Lender

A lender provides access to the superior technology $y(k;s)$, and a lending contract featuring a sequence of a state-contingent investments $k_t(h_t)$ and transfers $p_t(h_t)$ where $h_t \equiv (s_0, \ldots, s_t)$, provided the borrower has not defaulted at any previous date. As in the deterministic case, the analysis of Pareto efficient contracts with a minimum profit constraint π for the lender, reduces to the analysis of CE contracts with a lender breakeven constraint and a borrower initial wealth of $w = e - \pi$.

Such a contract generates welfare for an agent starting with wealth w and initial shock s_0:

$$u(w - k_0 + p_0) + E_{s_1,s_2,..}\Big[\sum_{t=1}^{\infty} \delta^t u(y(k_{t-1}, s_{t-1}) - k_t + p_t)\Big] \tag{17}$$

The lender's breakeven constraint is

$$E[\sum_{t=0}^{\infty} \delta^t p_t] \le 0 \tag{18}$$

while the incentive constraint requires at any date t and following any history h_t:

$$E\Big[\sum_{j=0}^{\infty} \delta^j u(y(k_{t-1+j}), s_{t+j}) - k_{t+j} + p_{t+j})|h_t\Big] \ge E_{s'}[V_A(y(k_{t-1}), s_t), s')] \tag{19}$$

A contract is feasible if it satisfies (18) and (19). Let welfare $V(w, s_0)$ denote the maximum value of (17) subject to the two feasibility constraints.

Analogous to the deterministic case, we obtain:

Lemma 9. *Any feasible contract can be implemented by a sequence of one period state-contingent loans satisfying* $l_0 = p_0$, $l_t(h_t) = p_t(h_t) - \frac{l_{t-1}(h_{t-1})}{\delta}$, *which (i) are always repaid on the equilibrium path, and (ii) satisfy the break-even condition:*

$$E[\lim_{T\to\infty} \delta^T l_T] \leq 0 \tag{20}$$

In other words, the lender provides a fresh one period loan at each date-history pair, which provides the required net transfer after allowing for repayment of the previous loan. The break-even condition (20) reduces to a 'no-Ponzi' scheme requirement which holds in expectation. To ensure this, we shall assume there is a finite lower bound $\underline{w}$ on the borrower's net wealth imposed by law or MFI policy. This amounts to a limit on loan size that depends on anticipated current output:

$$l_t \leq \delta[y(k_t, s_t) - \underline{w}] \tag{21}$$

As in the deterministic case, we shall ignore constraint (20) and then show that the solution to the 'relaxed' problem (which incorporates constraint (21) instead) satisfies it automatically

We then obtain a recursive representation of the optimal contracting problem:

$$V(w, s) = \max_{k,l}[u(w - k + l) + \delta E_{s'}[V(y(k, s) - \frac{l}{\delta}, s')] \tag{22}$$

subject to the incentive constraint

$$V(y(k, s) - \frac{l}{\delta}, s') \geq V_A(y(k, s), s'), \forall s' \in [\underline{s}, \bar{s}] \tag{23}$$

and borrowing constraint

$$l \leq \delta[y(k, s) - \underline{w}] \tag{24}$$

This is similar to the case of certainty, except that there is a separate incentive constraint for each possible realization of the shock at the next date, while the objective function involves only the corresponding expected value of the continuation utility.

The measure of net wealth is now $w \equiv y(k, s) - \frac{l}{\delta}$, which is bounded below by $\underline{w}$ and unbounded above. And we can continue to break down the recursive contracting problem into two steps. First, given current shock s and a target net wealth $\Omega \geq \underline{w}$ in the next period, minimize the cost in terms of foregone current consumption:

$$C(\Omega, s) \equiv \min(k - l) \quad \text{subject to: (23)} \quad \text{and} \quad y(k, s) - \frac{l}{\delta} = \Omega \tag{25}$$

Then at the second step, select the target wealth $\Omega(w,s)$ for next period:

$$V(w,s) = \max_{\Omega \geq \underline{w}} [u(w - C(\Omega, s)) + \delta E_{s'} V(\Omega, s')] \tag{26}$$

As in the deterministic case, optimal target wealths are well-defined, and we consider any policy function which is a measurable selection from the optimal policy correspondence. Let an optimal investment and financing policy be denoted by $k(w,s), l(w,s)$ respectively.

Our main result for the case of uncertainty is the following.

Proposition 3. *(a) Wealth next period $w' = \Omega(w;s)$ is non-decreasing in current wealth w and productivity shock s.*

(b) $k(w,s) \leq k_\delta(s)$ for all w, where $\delta y'(k_\delta(s);s) = 1$.

(c) Along any history, the no-Ponzi scheme condition $\lim_{T\to\infty} \delta^T l_T \leq 0$ holds, and the lender breaks even.

(d) $k(w;s)$ is non-decreasing in w.

(e) Conditional on $w_t \geq w_{t-1}$ and a sequence of nondecreasing productivity shocks:

$$s_{t+j} \geq s_{t+j-1}, \forall j = 0, 1, 2.. \tag{27}$$

net wealth w_{t+j} and investment k_{t+j} are nondecreasing in j.

(f) For any s, there exists wealth threshold $w^*(s)$ such that $k(w,s) = k_\delta(s)$ and $l(w,s) \leq 0$ for all $w \geq w^*(s)$.

Part (a) implies the evolution of wealth follows a monotone Markov process. (b) ensures there is never any under-investment, while (c) implies the break-even condition for the lender along every history. Part (d) says that investment is non-decreasing in wealth. These results imply part (e): conditional on a nondecreasing sequence of productivity shocks (combined with wealth rising initially), the wealth and investment of the borrower rises monotonically over time. This shows that the main result of Albuquerque and Hopenhayn (2004)) regarding the effect of rising 'age' of the relationship (conditional on productivity shock) continues to hold in this setting. Once wealth rises sufficiently, (f) states that investment levels become first-best. As wealth rises further, the agent becomes a lender rather than borrower and attains first-best investment. Of course, welfare is not first-best, owing to lack of insurance and resulting consumption distortions. Wealthy agents may suffer a string of negative productivity shocks and subsequent declines in wealth and forced to borrow again.

We have not provided any results concerning invariant or limiting wealth distributions. Such results could be obtained upon imposing exogenous upper bounds on lending and capital investment, as in (Hopenhayn and Prescott (1992)), which ensure a compact state space. Such bounds would be arbitrary and ad hoc, so we avoid imposing them. Whether such results can be obtained despite the absence of such bounds, remains an interesting open question.

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Table 1: MFI Portfolio at Risk Average, By Region

Regions[10]	Percentage of Total Borrowers	Percentage of Gross Loan Portfolio (GLP)	Portfolio at Risk> 30 Days (PAR)
Africa	5%	9%	10.60%
East Asia and the Pacific (EAP)	1%	16%	3.40%
Eastern Europe and Central Asia (ECA)	3%	11%	10.00%
Latin America and the Carribean (LAC)	19%	42%	5.40%
Middle East and North Africa (MENA)	2%	1%	3.60%
South Asia	57%	20%	2.60%

"Portfolio at Risk (PAR)": is one of the indicators of repayment rate. Portfolio at Risk [xx] days is defined as the value of all loans outstanding that have one or more installments of principal past due more than [xx] days. This includes the entire unpaid principal balance, including both the past due and future installments, but not accrued interest. It also includes loans that have been restructured or rescheduled.
"Gross Loan Portfolio (GLP)": All outstanding principals due for all outstanding client loans. This includes current, delinquent, and renegotiated loans, but not loans that have been written off.
Source MIX (2017): Global Outreach and Financial Performance Benchmark Report 2015.

Table 2: MFI Clienteles, Loan Portfolio and Progressive Lending Practices in Three Countries

Country	Number of Active Borrowers ′000	Gross Loan Portfolio (GLP)(USD)m	Financial Service Provider (FSP)	Number of Active Borrowers ′000	Gross Loan Portfolio (GLP)(USD)m	Progressive Lending?
India	38,097.6	11,640.8	Bandhan	-	2,352.66	Yes
			Janalakshmi	5,888.75	1,973.48	Yes
			Bharat Financial (SKS)	5,323.06	1,413.30	Yes
			Share	3,740.00	251.68	Yes
			SKDRDP	3,612.43	754.60	
Bangladesh	23,977.7	5,753.7	Grameen Bank	7,180.00	1,294.65	Yes
			ASA	6,207.69	1,533.97	Yes
			BRAC	5,356.52	1,768.61	
			BURO Bangladesh	917.46	311.61	Yes
			TMSS	736.98	188.95	Yes
Vietnam	7,533.9	7,351.9	VBSP	6,863.04	6,434.69	Yes
			CEP	288.49	108.28	Yes
			Coopbank	111.93	726.19	
			TYM	97.42	45.45	Yes
			MOM	40.17	7.26	

Top three countries by active borrowers and top five MFIs from each of them.
Source MIX (2017): Global Outreach and Financial Performance Benchmark Report 2015 and respective websites.

Appendix: Proofs

Proof of Lemma 1: Denote the original problem (1) by $\mathcal{P}(e,\pi)$ and note that it can be rewritten as follows with a change of variable from transfers p_t to consumption $c_t \equiv y(k_{t-1}) + p_t - k_t, t \geq 1$ and $c_0 \equiv e + p_0 - k_0$:

$$\max_{\left\langle \{c_t\}_{t=0}^{\infty}, \{k_t\}_{t=0}^{\infty} \right\rangle} V_0 \equiv \sum_{t=0}^{\infty} \delta^t u(c_t) \tag{28}$$

subject to:

$$\text{LPC}: \quad \sum_{t=0}^{\infty} \delta^t c_t \leq e - \pi - \sum_{t=0}^{\infty} \delta^t k_t + \sum_{t=1}^{\infty} \delta^t y(k_{t-1})$$

$$\text{IC}_t: \quad V_t \equiv \sum_{\tau=t}^{\infty} \delta^{\tau-t} u(c_\tau) \geq V^A(y(k_{t-1})), \quad \forall t \geq 1.$$

$$\text{BPC}: \quad V_0 \geq V^A(e).$$

Similarly refer to the CE problem (2) by $\mathcal{P}^C(w)$, and using the same change of variables this can be written as:

$$V(w) \equiv \max_{\left\langle \{c_t\}_{t=0}^{\infty}, \{k_t\}_{t=0}^{\infty} \right\rangle} \sum_{t=0}^{\infty} \delta^t u(c_t) \tag{29}$$

subject to:

$$\text{LBC}: \quad \sum_{t=0}^{\infty} \delta^t c_t \leq w - \sum_{t=0}^{\infty} \delta^t k_t + \sum_{t=1}^{\infty} \delta^t y(k_{t-1})$$

$$\text{IC}_t: \quad V_t \equiv \sum_{\tau=t}^{\infty} \delta^{\tau-t} u(c_\tau) \geq V^A(y(k_{t-1})), \quad \forall t \geq 1. \tag{30}$$

We claim that the feasible set in $\mathcal{P}(e,\pi)$ is non-empty if and only if the feasible set in $\mathcal{P}^C(e-\pi)$ is non-empty and $V(e-\pi) \geq V^A(e)$. To show the 'if' part, observe that the solution to $\mathcal{P}^C(e-\pi)$ is feasible in $\mathcal{P}(e,\pi)$. For the converse, take any feasible contract $\{\hat{c}_t, \hat{k}_t\}$ in $\mathcal{P}(e,\pi)$. This is feasible in $\mathcal{P}^C(e-\pi)$. Hence the solution to the latter problem yields a payoff $V(e-\pi)$ of at least $\sum_{t=0}^{\infty} \delta^t u(c_t)$, which in turn is at least $V^A(e)$ owing to (BPC).

Next observe that $V(.)$ is strictly increasing, since any increase in w can be accompanied by an equivalent increase in c_0 without violating either LBC or any IC_t.

Define $\bar{\pi}(e) \equiv \sup\{\pi | V(e-\pi) \geq V^A(e)\}$. It is evident that $\bar{\pi}(e) < \infty$, since $V^A(e) > 0$ for any $e > 0$. Also $\bar{\pi}(e) > 0$, since $V(e) > V^A(e)$, as the access to superior technology combined with financial autarky ensures a higher payoff to the borrower. Since $V(.)$ is strictly increasing, it follows that a feasible contract in $\mathcal{P}(e,\pi)$ exists if and only if $\pi \leq \bar{\pi}(e)$. Finally, observe that given any $\pi \leq \bar{\pi}(e)$, BPC is redundant in $\mathcal{P}(e,\pi)$, and the two problems $\mathcal{P}(e,\pi), \mathcal{P}^C(e-\pi)$ are the same as they involve the same objective function and constraint set. ■

Proof of Lemma 2: Already provided in the text.

Proof of Lemma 7: If this is false, there exist $w_1 < w_2$ with $\Omega_1 \equiv \Omega(w_1) > \Omega(w_2) \equiv \Omega_2$. Then $V(\Omega_1) > V(\Omega_2)$ and

$$u(w_2 - C(\Omega_2)) - u(w_2 - C(\Omega_1)) \geq \delta[V(\Omega_1) - V(\Omega_2)] > 0 \tag{31}$$

which implies $C(\Omega_1) > C(\Omega_2)$. On the other hand,

$$\delta[V(\Omega_1) - V(\Omega_2)] \geq u(w_1 - C(\Omega_2)) - u(w_1 - C(\Omega_1)) \tag{32}$$

which implies

$$u(w_2 - C(\Omega_2)) - u(w_2 - C(\Omega_1)) \geq u(w_1 - C(\Omega_2)) - u(w_1 - C(\Omega_1)) \tag{33}$$

This contradicts the concavity of u. ■

Proof of Lemma 8: Consider any w and a slightly higher wealth $w + \epsilon > w$. Since the incentive constraint in (9) does not depend on w, the policies $(k(w), l(w))$ and $(k(w+\epsilon), l(w+\epsilon))$ are feasible for both agents with starting wealth w and $w + \epsilon$. Therefore:

$$\begin{aligned} V(w+\epsilon) \equiv u(w+\epsilon - C(\Omega(w+\epsilon))) + \delta V(\Omega(w+\epsilon)) &\geq u(w+\epsilon - C(\Omega(w))) + \delta V(\Omega(w)) \\ V(w) \equiv u(w - C(\Omega(w))) + \delta V(\Omega(w)) &\geq u(w - C(\Omega(w+\epsilon))) + \delta V(\Omega(w+\epsilon)) \end{aligned}$$

which implies

$$\begin{aligned} \frac{u(w+\epsilon - C(\Omega(w+\epsilon))) - u(w - C(\Omega(w+\epsilon)))}{\epsilon} &\geq \frac{V(w+\epsilon) - V(w)}{\epsilon} \\ &\geq \frac{u(w+\epsilon - C(\Omega(w))) - u(w - C(\Omega(w)))}{\epsilon} \end{aligned}$$

Take limits as $\epsilon \to 0+$. Since Lemma 7 implies $C(\Omega(w))$ is nondecreasing in w, it is continuous almost everywhere. At any continuity point of $C(\Omega(w)$, it follows that the right-hand derivative of V exists and equals $u'(w - C(\Omega(w)))$. A parallel argument for the case of $\epsilon < 0$ with direction of inequalities reversed in (34) holds, implying the left-hand derivative of V also exists and equals $u'(w - C(\Omega(w)))$. Finally observe that for any $\epsilon > 0$, it is always feasible to let the agent consume the incremental wealth immediately, so right hand derivative of V is everywhere bounded below by $u'(c(w))$. ∎

Proof of Proposition 1: Consider any $w < w^*$. If $\Omega(w) \leq w$, Lemma 7 implies that starting from w the sequence of net wealth is monotonically nonincreasing. Conversely, if $\Omega(w) > w$, the sequence is monotonically nondecreasing. Hence either way, the sequence of net wealths must converge. This implies that the sequence of consumption and investments must also converge.

Next we show that the limiting wealth w_∞ cannot be smaller than w^*. Suppose otherwise. Then we claim there is a variation on the contract which is feasible and raises the borrowers welfare. Since $w_\infty < w^*$, for all large t we have $w_t < w^*$, and there is underinvestment in the limit ($k_\delta > k_\infty$). So $[\delta y'(k_t) - 1]$ is positive and bounded away from zero for all large t.

For all large t, the incentive constraint binds, hence $V(y(k_t) - \frac{l_t}{\delta}) = V^A(y(k_t))$. Consider an increase in k_t by $\epsilon > 0$, and let l_t change by $\Delta_t \epsilon$ where

$$\Delta_t = \delta y'(k_t)[1 - \frac{V^{A'}(y(k_t))}{u'(c_{t+1})}] \tag{34}$$

For ϵ sufficiently small, the IC is preserved because:

$$\begin{aligned} & \frac{\partial V(y(k_t + \epsilon) - \frac{l_t}{\delta} - \frac{\Delta_t}{\delta}\epsilon)}{\partial \epsilon}|_{\epsilon=0+} \\ \geq \quad & u'(c_{t+1})[y'(k_t) - \frac{\Delta_t}{\delta}] \\ = \quad & V^{A'}(y(k_t))y'(k_t) \end{aligned} \tag{35}$$

where the last equality follows from construction of Δ_t (equation (34)), and the preceding inequality follows from Lemma 8.

The resulting borrower's welfare at t is

$$V_t(\epsilon) \equiv u(w_t + l_t + \Delta_t\epsilon - k_t - \epsilon) + \delta V(y(k_t + \epsilon) - \frac{l_t + \Delta_t\epsilon}{\delta}) \tag{36}$$

implying that at $\epsilon = 0$, the rate of rise of V_t is at least:

$$\begin{aligned} & -u'(c_t)[1-\Delta_t] + u'(c_{t+1})[\delta y'(k_t) - \Delta_t] \\ = \quad & u'(c_t)[\delta y'(k_t) - 1] + [u'(c_{t+1}) - u'(c_t)][\delta y'(k_t) - \Delta_t] \end{aligned} \tag{37}$$

For t sufficiently large (37) is positive, because $[\delta y'(k_t) - 1]$ is positive and bounded away from zero, $[u'(c_{t+1}) - u'(c_t)]$ converges to zero, and $[\delta y'(k_t) - \Delta_t)] = \delta y'(k_t)\frac{V^{A'}(y(k_t))}{u'(c_{t+1})}$ is positive and converges to a finite number as $t \to \infty$.

Hence $w_\infty \geq w^*$. Since $\Omega(w^*) = w^*$, Lemma 7 implies that $\Omega(w) \leq w^*$ for any $w < w^*$. Hence $w_\infty \leq w^*$, and it follows that $w_\infty = w^*$.

Next, we show that first-best wealth w^* cannot be achieved at any finite date. Otherwise, there exists some date t with $w_t < w^*$ and $\Omega(w_t) = w_{t+1} = w^*$. From Lemma 6 it follows that $k_(w_t) = k_\delta$, and hence $\delta y'(k_t) = 1$, while $c_{t+1} = c(w_{t+1}) = c^*(w^*)$. And c_t must be strictly lower than $c^*(w^*)$, otherwise the agent achieves welfare at least $V(w^*)$ at a wealth $w_t < w^*$. So there is a consumption distortion resulting in $u'(c_{t+1}) - u'(c_t) < 0$. Now we can consider a sequence of reverse perturbations analogous to that constructed above, with $\epsilon < 0$ and converging to 0 from below. From (37) it is evident that this will raise welfare for ϵ close enough to zero.

Finally, the sequence of wealths must be strictly increasing at every date (otherwise $w_{t+1} = \Omega(w_t) = w_t$, and wealth will remain at $w_t < w^*$ for ever). ∎

Proof of Proposition 2: Whenever $w < w^*$ the IC binds, hence $V(\Omega(w)) \equiv V(y(w) - \frac{l(w)}{\delta}) = V^A(y(w))$ implies $\Omega(w) < y(w)$ since $V(w') > V^A(w')$ for all w'. Hence $l(w) \equiv \delta[y(w) - \Omega(w)] > 0$.
Next, we show that $l(w)$ must be strictly increasing for all $w < w^*$.
Claim 1: To establish this, it suffices to show that $c(\Omega(w)) < c^A(y(w))$ for all $w < w^*$, i.e., optimal consumption on the equilibrium path is smaller than the optimal consumption off the equilibrium path in the first period of any deviation. This is because the right hand derivative of V at $\Omega(w)$ is bounded below by $u'(c(\Omega(w)))$, while the slope of $V^A(y(w))$ equals $u'(c^A(y))$ (where $c(y), c^A(y)$ respectively denote the optimal consumptions at the first date along the equilibrium path and in autarky respectively, starting with wealth y). Hence $c(\Omega(w)) < c^A(y(w))$ implies the slope of V at $\Omega(w)$ is larger than $V^{A'}(y(w))$. So $\Omega(w)$ must rise more slowly in w than $y(w)$.

Next, observe that if $k(w) \geq k^A_\delta$, off-equilibrium-path consumption is stationary. Then Claim 1 follows from the IC: $V(\Omega(w)) = V^A(y(w))$, since consumption is growing on the equilibrium path (an argument similar to that used in Proposition 1 rules out the possibility that consumption is stationary on the equilibrium path).

Let $c_{t'}^{A,t}$ denote optimal consumption at date t' in autarky, resulting from a deviation at $t < t'$.

Claim 2: If at any date t: $c_t^{A,t} \leq c_t$, then the same must be true at $t+1$: $c_{t+1}^{A,t+1} \leq c_{t+1}$.

To establish Claim 2, suppose otherwise that at some date t: $c_t^{A,t} \leq c_t$ and $c_{t+1}^{A,t+1} > c_{t+1}$. Then

$$\Delta_t \equiv \delta y'(k_t)[1 - \frac{u'(c_{t+1}^{A,t+1})}{u'(c_{t+1})}] > 0$$

and so

$$u'(c_t)[1 - \Delta_t] < u'(c_t) \leq u'(c_t^{A,t}) = \delta \frac{\partial}{\partial k}[V^A(g(k_t^{A,t}))] \tag{38}$$

Since the IC at t binds, we have

$$u(c_t) + \delta V(y_{t+1} - \frac{l_t}{\delta}) = u(c_t^{A,t}) + \delta V^A(g(k_t^{A,t}))$$

so $c_t^{A,t} \leq c_t$ implies

$$V^A(g(k_t^{AA,t})) \geq V(y_{t+1} - \frac{l_t}{\delta}) \geq V^A(y_{t+1})$$

where the last inequality again uses the IC at t. Hence $g(k_t^{A,t}) \geq y_{t+1}$. Since V^A is concave, this implies $\delta \frac{\partial}{\partial k}[V^A(g(k_t^{A,t}))] \leq \delta \frac{\partial}{\partial k}[V^A(y_{t+1})]$. Hence (38) implies

$$u'(c_t)[1 - \Delta_t] < \delta \frac{\partial}{\partial k}[V^A(y_{t+1})]$$

Using an argument similar to that used in Proposition 1 , it is feasible to increase investment slightly on the equilibrium path and raise the borrower's welfare, contradicting optimality of the original contract. Hence Claim 2 holds.

Finally, Claim 2 implies by induction that $c_{t'}^{A,t'} \leq c_{t'}$ for all $t' > t$. Since k_t converges to the first-best capital stock k_δ which strictly exceeds k_δ^A, there exists some date $t' > t$ when $k_{t'} = k(w_{t'}) > k_\delta^A$ and we would obtain a contradiction to the argument in the previous paragraph. This completes the proof. ∎

Proof of Proposition 3:

(a) The same argument as in the deterministic case ensures $\Omega(w; s)$ is non-decreasing in w. To show that it is non-decreasing in s, it suffices to show that C and marginal cost of wealth $C_w(\Omega, s)$ are both non-increasing in s.

Observe that

$$C(\Omega, s) = \delta\Omega - \delta Z(\Omega, s) \tag{39}$$

where

$$Z(\Omega, s) \equiv \max_{y \geq 0}[y - \frac{k(y,s)}{\delta}] \qquad \text{subject to:} \qquad V_A(y, s') \leq V(\Omega, s') \forall s' \tag{40}$$

and $k(Y, s)$ denotes the solution for k in $y(k, s) = Y$.

Since (using standard arguments) $V_A(y, s')$ is strictly increasing and continuous in y, we can define $Y(\Omega)$ as the largest (or supremum) y satisfying the incentive constraint (IC) in (40). The IC can then be replaced by $y \leq Y(\Omega)$. Standard arguments also imply $V(w, s)$ is increasing in w, hence $Y(\Omega)$ is increasing. Therefore

$$Z(\Omega, s) \equiv \max_{y \geq 0}[y - \frac{k(y,s)}{\delta}] \qquad \text{subject to:} \qquad y \leq Y(\Omega) \tag{41}$$

As $k(y, s)$ is non-increasing in s, it follows that Z is non-decreasing in s, and C is non-increasing in s.

The unconstrained solution to (41) involves setting $y(\Omega, s) = y_\delta(s) \equiv y(k_\delta(s), s))$, and the IC binds iff $y_\delta(s) > Y(\Omega)$. Hence $y(\Omega, s) = Y(\Omega)$ if $y_\delta(s) > Y(\Omega)$ and $y_\delta(s)$ otherwise. It follows that $C(\Omega, s) = \delta\Omega - y(k_\delta(s), s)) + \frac{1}{\delta} y(k_\delta(s), s))$ in the former case, and $\delta\Omega - Y(\Omega, s) + \frac{1}{\delta} k(Y(\Omega, s), s)$ otherwise.

Hence the marginal cost of a higher wealth target $C_w(\Omega, s)$ equals δ if Ω is large enough that IC does not bind, so does not depend on s. When the IC does bind, the marginal cost is defined a.e. (whenever $Y'(\Omega)$ exists), in which case it equals $\delta - Y'(\Omega)[1 - \frac{1}{\delta} y_k(k(Y(\Omega), s), s)]$. This is non-increasing in s because $y_k(k, s)$ is decreasing in k and increasing in s, $k(Y, s)$ is decreasing in s, and $Y'(\Omega) > 0$.

(b) Suppose $\delta y'(k(w, s)) < 1$. Consider a small reduction in both capital and borrowing: $k = k(w, s) - \epsilon$ and $l = l(w, s) - \epsilon$, which then results in an increase in $y(k, s) - \frac{l}{\delta}$. The IC constraint (23) and borrowing constraint (24) continue to hold, continuation utility $E_{s'}[V(y(k, s) - \frac{l}{\delta}), s']$ rises and current consumption is unchanged. So the original contract could not have been optimal.

(c) From (b) and (24) we have $l_t(w, s) \leq \delta[y_\delta(s) - \underline{w}] \leq \delta \max_s[y_\delta(s) - \underline{w}]$, so the size of loans is uniformly bounded above, implying $\lim_{T \to \infty} \delta^T l_T \leq 0$ along any history.

(d) If this is false, there exists s and wealths $w > w'$ with $k(w, s) < k(w', s) \leq k_\delta(s)$. From (a), $y(k(w, s), s) - \frac{l(w,s)}{\delta} \geq y(k(w', s), s) - \frac{l(w',s)}{\delta}$, which implies

$$l(w', s) - l(w, s) \geq \delta[y(k(w', s), s) - y(k(w, s), s)] \tag{42}$$

Define $l'' \equiv [k(w', s) - k(w, s)] + l(w, s)$. Since $k(w, s) < k(w', s) \leq k_\delta(s)$, the Intermediate Value Theorem implies

$$\delta[y(k(w', s), s) - y(k(w, s), s)] > k(w', s) - k(w, s) = l'' - l(w, s). \tag{43}$$

(42) then implies that $l(w', s) > l''$. Therefore:

$$y(k(w', s), s) - \frac{l''}{\delta} > y(k(w', s), s) - \frac{l(w', s)}{\delta} \tag{44}$$

implying that the agent with wealth w could feasibly borrow l'' and invest $k(w', s)$ while preserving the incentive and borrowing constraints. This would entail the same current consumption: $k(w, s) - l(w, s) = k(w', s) - l''$, and generate higher continuation utility because of (43), so we obtain a contradiction.

(e) Note to start with that $w_{t+1} = \Omega(w_t, s_t) \geq w_t = \Omega(w_{t-1}, s_{t-1})$ upon using (a), combined with $w_t \geq w_{t-1}$ and $s_t \geq s_{t-1}$. Now use the same argument inductively at all subsequent dates.

(f) If this is false, there is some s for which $k(w, s) < k_\delta(s)$ for all w. Observe first that this implies $l(w, s) \geq 0$ for all w, since otherwise $l(w, s) < 0$ for some w and the IC (23) does not bind for an agent with wealth w. This agent can increase k and l by the same small amount and raise continuation utility strictly, while keeping current consumption unchanged. Since $y'(k(w, s), s) > \frac{1}{\delta} > 1$, the borrowing constraint is also preserved. So the contract could not have been optimal.

Since $l(w, s) \geq 0$ for all w, (b) implies that net wealth next period $\Omega(w, s) \equiv y(k(w, s), s)) - \frac{l(w,s)}{\delta} \leq y_\delta(s)$ for all w, i.e., is bounded above. Consumption in the next period is therefore bounded above because investment is nonnegative and borrowing is bounded above by $\max_{s'} y_\delta(s') - \underline{w}$.

On the other hand, current consumption for the agent is bounded below by $w - k_\delta(s)$ since borrowing is non-negative for all w. Hence as $w \to \infty$, current consumption goes to ∞, while consumption in the next period is bounded above. Since the utility function satisfies Inada conditions, the marginal utility of current consumption must be lower than discounted marginal utility of consumption in the following period, for sufficiently large w. Such an agent can reduce l slightly below $l(w, s)$ which preserves borrowing and incentive constraints, and raises welfare. So the contract cannot be optimal for large enough w. ∎